

MODEL THEORY AND PROOF THEORY OF CPL

TADEUSZ LITAK, DIRK PATTINSON, KATSUHIKO SANO, AND LUTZ SCHRÖDER

Friedrich-Alexander-Universität Erlangen-Nürnberg
e-mail address: tadeusz.litak@fau.de

Australian National University
e-mail address: dirk.pattinson@anu.edu.au

Graduate School of Letters, Hokkaido University
e-mail address: v-sano@let.hokudai.ac.jp

Friedrich-Alexander-Universität Erlangen-Nürnberg
e-mail address: Lutz.Schroeder@cs.fau.de

ABSTRACT. We propose a generalization of first-order logic originating in a neglected work by C.C. Chang: a natural and generic correspondence language for any types of structures which can be recast as Set-coalgebras. We discuss axiomatization and completeness results for several natural classes of such logics. Moreover, we show that an entirely general completeness result is not possible. We study the expressive power of our language, both in comparison with coalgebraic hybrid logics and with existing first-order proposals for special classes of Set-coalgebras (apart from relational structures, also neighbourhood frames and topological spaces). Basic model-theoretic constructions and results, in particular ultraproducts, obtain for the two classes that allow completeness—and in some cases beyond that. Finally, we discuss a basic sequent system, for which we establish a syntactic cut-elimination result.

1. INTRODUCTION

Modal logics are traditionally a core formalism in computer science. Classically, their semantics is relational, i.e. a model typically comes with a set of states and one or several binary accessibility relations on the state set. However, non-relational semantics of various descriptions have come to play an increasing role, e.g. in concurrency, reasoning about knowledge and agency, description logics and ontologies: Models may involve such diverse features as concurrent games, as in coalition logic and alternating-time temporal logic [AHK02, Pau02]; probabilities [LS91, FH94, HM01]; integer weights as in the multigraph semantics of graded modal logic [DV02]; neighbourhoods [Che80]; and selection functions or preference orderings as in the different variants of conditional logic [Lew73, Che80]. *Coalgebraic modal logic* serves as a unifying framework for such non-relational modal logics [CKP⁺11].

Relational modal logic can be seen as a subset of first-order logic, specifically as the bisimulation-invariant fragment as shown by van Benthem for arbitrary models and later

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shown for finite models by Rosen [vB76, Ros97]. An analogous first-order counterpart for coalgebraic modal logic has been introduced in previous work by two of the authors [SP10a]. The language described there does support a van Benthem/Rosen-style theorem. It is quite expressive but has a fairly complex syntax with three sorts, modelling states, sets of states, and composite states, respectively, and is equipped with a carefully tuned Henkin-style semantics. In the current work we develop *coalgebraic predicate logic (CPL)*, a first-order correspondence language for coalgebraic predicate logic that is slightly less expressive than the language proposed originally but has a simpler syntax and a straightforward semantics that does not require any design decisions. The naturality of CPL is further corroborated by the fact that CPL is expressively equivalent to hybrid logic (see the overview article by Areces and ten Cate [AtC07]) with satisfaction operators and universal quantification (equivalently with the downarrow binder \downarrow and a global modality). Thus, CPL not only serves as a correspondence language for coalgebraic modal logic but also arises by adding a standard set of desirable expressive features widely used in specification and knowledge representation.

Our proposal originates in a largely forgotten paper by C.C. Chang [Cha73] who introduces a first-order logic of *Scott-Montague neighbourhood frames*, which in coalgebraic terms can be seen as coalgebras for the doubly contravariant powerset functor. Chang’s original motivation was to simplify model theory for what Montague called *pragmatics* and to replace Montague’s many-sorted setting by a single-sorted one. Chang’s contributions were primarily of a model-theoretic nature. He provided adaptations of the notions of (elementary) submodel/extension, elementary chain of models and ultraproduct and established a Tarski-Vaught theorem as well as downward and upward Löwenheim-Skolem theorems. Our syntax is a notational variant of Chang’s syntax; semantically, we generalize from neighbourhood frames to coalgebras for an underlying set functor, thus capturing the full range of non-relational modalities indicated above.

Our semantics naturally extends coalgebraic modal logic in that it is parametrized over an interpretation of the modal operators as predicate liftings [Pat03, Sch08]. It can thus be instantiated with modalities such as, for instance, the standard relational \Diamond ; with neighbourhood-based modalities as in Chang’s original setup; with probabilistic operators L_p ‘with probability at least p ’; or with a binary conditional \Rightarrow ‘if – then normally’. We incorporate such modalities \heartsuit into a first-order language by allowing formulas of the form (in case \heartsuit is unary)

$$t\heartsuit[z : \phi]$$

where t is a term, ϕ is a formula of coalgebraic predicate logic and z is a (comprehension) variable. Such a formula stipulates that t satisfies \heartsuit , applied to the set of all z that satisfy ϕ . For example, in standard modal logic over relational semantics, the formula $x\Diamond[z : z = y]$ says that x has y as a (relational) successor. In the probabilistic setting, the formula $xL_p[y : y \neq x]$ states that the probability of moving from x to a different state is at least p .

As indicated above, CPL supports a van Benthem / Rosen type result stating essentially that coalgebraic modal logic is the bisimulation-invariant fragment of CPL both over the class of all structures and over the class of finite structures; this result is proved in a companion paper [SPLar], which also establishes a Gaifman-type theorem for CPL. In the current paper, we establish the following results on CPL:

- We give a Hilbert-style axiomatization that we prove strongly complete for two particular classes of coalgebraic structures, viz. structures that are either

neighbourhood-like or *bounded*, where the latter type includes the relational and the graded case as well as positive Presburger modalities.

- While boundedness is a rather strong condition on structures, we show that the condition is fairly essential for completeness in the sense that within a much broader type of ω -*bounded* structures, the bounded structures are the only ones that allow for strong completeness.
- As indicated above, we establish the equivalence of CPL and several natural variants of coalgebraic hybrid logic.
- We prove some basic model-theoretic results. Specifically, we show that, under the same (alternative) assumptions as for our completeness result, ultraproducts exist and a downward Löwenheim-Skolem theorem holds; in fact, it turns out that the latter is applicable more broadly, requiring as it does only ω -boundedness in place of boundedness in its corresponding variant.
- We give sequent systems complementing the above-mentioned Hilbert system, and establish completeness, under the same (alternative) assumptions as for the Hilbert system, and more interestingly, syntactic cut-elimination for the “neighbourhood-like” case.

The material is organized as follows. In §2 we introduce the syntax and semantics of CPL and give a number of intuitive examples. In §3 we discuss the Hilbert-style axiomatization and associated completeness results. We proceed to clarify the relationship between CPL and several variants of coalgebraic modal and hybrid logic in §4. In §5 we take first steps in the model theory of CPL, and §6 deals with proof theory.

Further Related Work. As already discussed, the syntax of our logic follows Chang’s first-order logic of neighbourhood frames [Cha73]. An alternative, two-sorted language for neighbourhood frames has been proposed by Hansen et al. [HKP09]. Over neighbourhood frames, the language studied in the present work is a fragment of the two-sorted one; we give details in § 2.

First-order formalisms have also been considered for topological spaces, which are particular instances of neighbourhood frames when defined in terms of local neighbourhood bases. In particular, Sgro [Sgr80] studies interior operator logic in topology with interior modalities for finite topological powers of the space. This language is the weakest one in the hierarchy of topological languages considered in an early overview by Ziegler [Zie85]. Makowsky and Marcja [MM77] prove a range of completeness theorems for topological logics, including a completeness result for the Chang language itself, i.e., a special version of our Theorem 3.15. See also ten Cate et al. [CGS09] for a more contemporary reference. Despite the fact that CPL combines quantifiers and modalities, it should not be confused with what is usually termed quantified or first-order modal logic; see Remark 2.1.

As mentioned above, our logic is less expressive but more naturally defined than the correspondence language used in the first van Benthem/Rosen type characterization result for coalgebraic modal logic [SP10a]. Axiomatizations and model-theoretic results as we develop here are not currently available for the more expressive language of [SP10a].

A different generic first-order logic largely concerned with the Kleisli category of a monad rather than with coalgebras for a functor is introduced and studied in [Jac10]. Of all the languages discussed above, this one seems least related to the present one; indeed, the study of connections with languages like that of the original, three-sorted variant [SP10a] is mentioned by Jacobs [Jac10] as a subject for future research.

This paper is based on results first announced in earlier conference papers [LPSS12, LPS13]. Compared to the conference versions, it features full proofs and additional examples. Some results previously only mentioned such as the Omitting Types Theorem (Theorem 3.25) are explicitly stated and proved here for the first time. We also corrected a number of errors and typos. Most notably, as reconstructing the proof of cut-elimination for the G3c-style system proposed in [LPS13] proved problematic, we replaced it with a G1c-style system in this version, with a different treatment of equality and provided all the proof details.

2. SYNTAX, SEMANTICS AND EXAMPLES

We proceed to give a formal definition of *coalgebraic predicate logic (CPL)*. We fix a set Σ of predicate symbols and a modal similarity type Λ , i.e. a set of modal operators. Modal operators $\heartsuit \in \Lambda$ and predicate symbols $P \in \Sigma$ both come with fixed *arities* $\text{ar}(\heartsuit), \text{ar}(P) \in \mathbb{N}$. The set $\text{CPL}(\Lambda, \Sigma)$ of CPL *formulas* over Λ and Σ is given by the grammar

$$\text{CPL}(\Lambda, \Sigma) \ni \phi, \psi ::= y_1 = y_2 \mid P(\vec{x}) \mid \perp \mid \phi \rightarrow \psi \mid \forall x. \phi \mid x \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n]$$

where $\heartsuit \in \Lambda$ is an n -ary modal operator and $P \in \Sigma$ a k -ary predicate symbol, x, y_i are variables from a fixed set iVar we keep implicit. We just write $\text{CPL}(\Lambda)$ for $\text{CPL}(\Lambda, \emptyset)$. Booleans and the existential quantifier are defined in the standard way. We do not include function symbols, which can be added in a standard way [Cha73]. In the $[y_i : \phi_i]$ component, y_i is used as a comprehension variable, i.e., $[y_i : \phi_i]$ denotes a subset of the carrier of the model, to which modal operators can be applied in the usual way. In $x \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n]$, x is free and y_i is bound in ϕ_i , otherwise the notions of freeness and boundedness are standard. A variable is *fresh* for a formula if it does not have free occurrences in it; to save space, we will also sometimes say that $x \in \text{iVar}$ is fresh for $y \in \text{iVar}$ whenever it is distinct from it. A *sentence* or a *closed formula*, as usual, is a formula without free variables; otherwise, we speak of *open formulas*.

As usual, some care is needed when defining substitution to avoid, on the one hand, capture of newly substituted variables by quantifiers and on the other hand, substituting for a bound variable. We take as our model the discussion in Enderton's monograph [End01, p. 112–113]. As we now have two ways in which a variable can become bound and the binder \heartsuit involves also a variable/term in a non-binding way, it is desirable to spell out details. We thus define—prima facie not necessarily capture-avoiding—substitution $\alpha[t/x]$ with $t, x \in \text{iVar}$ (had we allowed for function symbols, t could be any term) as replacing x with t in atomic formulas and commuting with implication (and of course other boolean connectives, were they taken as primitives). For binders, the clauses are:

$$\begin{aligned} (\forall x. \phi)[t/y] &= \begin{cases} \forall x. \phi & x = y \\ \forall x. \phi[t/y] & \text{otherwise,} \end{cases} \\ (x \heartsuit [z_1 : \phi_1] \dots [z_n : \phi_n])[t/y] &= u \heartsuit [z_1 : \phi'_1] \dots [z_n : \phi'_n] \\ \text{where } \phi'_i &= \begin{cases} \phi_i & y = z_i \\ \phi_i[t/y] & \text{otherwise,} \end{cases} \quad \text{and } u = \begin{cases} t & x = y \\ x & \text{otherwise.} \end{cases} \end{aligned}$$

This of course cannot work without restrictions, so we follow Enderton in defining the notion of *substitutability* of t for x in a term. There are no restrictions on substitutability in atomic formulas, and for implications, it is defined as substitutability in the two argument

formulas. Finally, t is substitutable for x in $z\heartsuit[y_1 : \phi_1] \dots [y_n : \phi_n]$ whenever for every i , t is either

- fresh for $[y_i : \phi_i]$ (this includes the case $t = y_i$) or
- different from y_i and substitutable for x in ϕ_i .

Note that in the first alternative, substituting t for x has no effect on $[y_i : \phi_i]$. In a language with more general terms, the second alternative would require that y_i is fresh for t rather than different from t . Substitutability of t for x in $\forall y_1.\phi_1$ is defined similarly (and standardly).

We depart from Enderton's conventions by restricting, from now on, the usage of the $\alpha[t/x]$ notation to the case where t is substitutable for x in α (as usual, when this is not the case the substitution can still be applied after suitably renaming bound variables in α). For example, the axiom scheme $\forall \vec{y}.(\forall x.\phi \rightarrow \phi[z/x])$ denoted as EN2 in Table 1 has as its valid instances only those formulas where z is substitutable for x .

The semantics of CPL is parametrized over the choice of an endofunctor T on **Set** that determines the underlying system type: models are based on T -coalgebras, i.e. pairs $(C, \gamma : C \rightarrow TC)$ consisting of a carrier set C of worlds or *states* and a *transition function* γ . We think of the elements of TC as being *composite states*; e.g. if T is the identity functor then a composite state is just a state, and if T is powerset, then a composite state is a set of states. Thus, the transition function assigns to each state c a composite state $\gamma(c)$ that represents the successors of c and that we correspondingly refer to as the *composite successor* of c . E.g. in case T is powerset, a T -coalgebra assigns to each state a set of successor states, and hence is essentially a Kripke frame.

To interpret the modal operators, we extend T to a Λ -structure, i.e. we associate to every n -ary modal operator $\heartsuit \in \Lambda$ a set-indexed family of mappings

$$\llbracket \heartsuit \rrbracket_C : (\mathcal{Q}C)^n \rightarrow \mathcal{Q}TC$$

where \mathcal{Q} denotes the contravariant powerset functor, subject to *naturality*, i.e.

$$(Tf)^{-1} \circ \llbracket \heartsuit \rrbracket_C = \llbracket \heartsuit \rrbracket_D \circ (f^{-1})^n$$

for every set-theoretic function $f : C \rightarrow D$. In categorical parlance, this means that $\llbracket \heartsuit \rrbracket$ is a natural transformation $\mathcal{Q}^n \rightarrow \mathcal{Q} \circ T^n$; we recall that the contravariant powerset functor \mathcal{Q} maps a set X to the powerset of X and a map $X \rightarrow Y$ to the preimage map $\mathcal{Q}f : \mathcal{Q}Y \rightarrow \mathcal{Q}X$, i.e. $\mathcal{Q}f(A) = f^{-1}[A]$ for $A \subseteq Y$. Formally speaking, we should define a Λ -structure as a pair $(T, \{\llbracket \heartsuit \rrbracket\}_{\heartsuit \in \Lambda})$, but to avoid cumbersome notation and terminology, we will speak about a Λ -structure *based on* T (or a Λ -structure *over* T) and suppress the second element of the pair whenever $\{\llbracket \heartsuit \rrbracket\}_{\heartsuit \in \Lambda}$ is clear from the context.

A triple $\mathfrak{M} = (C, \gamma, I)$ consisting of a coalgebra $\gamma : C \rightarrow TC$ and a predicate interpretation $I : \Sigma \rightarrow \bigcup_{n \in \omega} \mathcal{Q}(C^n)$ respecting arities of symbols will be called a (*coalgebraic*) *model*.

In other words, a coalgebraic model consists simply of a **Set**-coalgebra and an ordinary first-order model whose universe coincides with the carrier of the coalgebra. Given a model $\mathfrak{M} = (C, \gamma, I)$ and a valuation $v : \text{iVar} \rightarrow C$, we define satisfaction $\mathfrak{M}, v \models \phi$ in the standard way for first-order connectives and for \heartsuit by the clause

$$\mathfrak{M}, v \models x\heartsuit[y_1 : \phi_1] \dots [y_n : \phi_n] \iff \gamma(v(x)) \in \llbracket \heartsuit \rrbracket_C(\llbracket \phi_1 \rrbracket_C^{y_1}, \dots, \llbracket \phi_n \rrbracket_C^{y_n})$$

where $\llbracket \phi \rrbracket_C^y = \{c \in C \mid \mathfrak{M}, v[c/y] \models \phi\}$ and $v[c/y]$ is v modified by mapping y to c .

Remark 2.1. *Quantified* or *first-order modal logic* in the sense used widely in the literature (see, e.g., [Gar01]) combines quantification and modalities in a two-sorted and,

effectively, two-dimensional semantics: One has an underlying set of worlds as well as an underlying set of individuals, with modalities interpreted as moving between worlds and quantification interpreted as ranging over individuals in the current world. We emphasize that although CPL also combines modalities and quantifiers, it is not a quantified modal logic in this sense: it is interpreted over a single set of individuals, and both the modalities and the quantifiers move within this set. In particular, the instance of CPL induced by the standard modalities equipped with their usual predicate liftings is standard first-order logic rather than a quantified modal logic, as we discuss below in some detail.

In a companion paper on the van Benthem-Rosen theorem for CPL [SPLar] and in conference papers the present work is based upon [LPSS12, LPS13], we have focused on Chang’s original motivation for this language [Cha73]. Namely, Chang saw his setup as a modification of Montague’s account of *pragmatics*, tailored to reasoning about social situations and relationships between an individual and sets of individuals. We have proposed a series of examples kept in the same spirit, utilizing Facebook, Twitter and social networks. In the present paper, we offer examples based on, so to say, networking of a more low-level character, especially *delay-* or *disruption-tolerant* networking (*DTN*). We do not claim to be very accurate with respect to specifications of concrete protocols; our examples are of purely inspirational and illustrative character. It is worth mentioning, though, that such routing and forwarding protocols can be backed by social insights [HCY08], so in a sense we are still following the spirit of our original examples.¹

Neighbourhood Frames. Scott/Montague *neighbourhood semantics* is captured coalgebraically using $\Lambda = \{\Box\}$ and putting $TC = \mathcal{Q}QC$ (the doubly contravariant powerset functor), which extends to a Λ -structure by

$$\llbracket \Box \rrbracket_C(A) = \{\sigma \in TC \mid A \in \sigma\}.$$

A T -coalgebra then associates to each state a set of sets of states, i.e. a system of neighbourhoods; thus, T -coalgebras are just neighbourhood frames. In the presence of a binary relation $S(x, y)$ that we read as ‘node/router y is in the forwarding table of x ’ and interpreting \Box as ‘is a recognized subcommunity’, the formula

$$\exists y. x \Box [z : S(z, y)]$$

reads as ‘there exists a certain y such that amongst the subcommunities recognized by x , there is one formed exactly by those having y in its forwarding table’.

The instance of CPL that we obtain in this way is, up to quite minor syntactic differences, Chang’s original language [Cha73]. As mentioned in § 1, it embeds as a fragment into Hansen et al.’s two-sorted correspondence language [HKP09]. We refrain from giving full syntactic details; roughly, the setup is as follows: The two-sorted language has sorts s for states and n for neighbourhoods, and features binary infix predicates **N** and **E** respectively modelling the neighbourhood relation between states and neighbourhoods, and the inverse elementhood relation \ni between neighbourhoods and states. Then our $x \heartsuit [y : \phi(y)]$ can be translated as $\exists u. (xNu \wedge \forall y. (uEy \leftrightarrow \phi(y)))$.

¹In particular, Hui et al. [HCY08] gave us the idea of using *subcommunities* in this context.

Relational first-order logic. Instantiating CPL with the usual modalities of relational modal logic, specifically the logic K , we obtain a notational variant of ordinary FOL over relational structures, that is, of the usual correspondence language. The main idea has already been indicated in the introduction: we encode the successor relation in formulas of the form $x\Diamond[z : y = z]$, which state that y is a successor of x . Formally, we capture the standard modality and the propositional atoms of the relational modal logic K in the similarity type

$$\Lambda = \{\Diamond\} \cup \text{At}$$

where At is a set of propositional atoms; as expected, \Diamond is unary, and $a \in \text{At}$ is nullary. We interpret these operators over the functor T given on objects by

$$TX = \mathcal{P}X \times \mathcal{P}\text{At}$$

where \mathcal{P} denotes the covariant powerset functor. That is, a coalgebra $\gamma : C \rightarrow TC$ assigns to each state $c \in C$ a set of successors as well as a set of propositional atoms valid in c . The interpretation is defined by means of predicate liftings

$$\begin{aligned} \llbracket \Diamond \rrbracket_X(A) &= \{(Y, U) \in \mathcal{P}X \times \mathcal{P}\text{At} \mid A \cap Y \neq \emptyset\} \\ \llbracket a \rrbracket_X &= \{(Y, U) \in \mathcal{P}X \times \mathcal{P}\text{At} \mid a \in U\} \end{aligned}$$

where, corresponding to the arity of the modal operators, the predicate lifting for \Diamond is unary and the predicate liftings for the $a \in \text{At}$ are nullary. These predicate liftings capture precisely the standard semantics of both \Diamond and the propositional atoms. In particular, the above-mentioned formula $x\Diamond[z : y = z]$ really does say that y is a successor of x . (Notice that in the nullary case, our syntax instantiates to formulas xa saying that x satisfies the propositional atom a).

The standard first-order correspondence language of modal logic has unary predicates a for the atoms $a \in \text{At}$ and a binary predicate R to represent the successor relation. We translate $\text{CPL}(\Lambda)$ as defined above into the standard correspondence language by just extending the standard translation of modal logic to CPL, with the modification that the current state is represented by an explicit variable in CPL so that it is no longer necessary to index the standard translation with a variable name. That is, our translation ST is defined in the modal cases (which by our conventions include the case of propositional atoms) by

$$\begin{aligned} ST(x\Diamond[y : \phi]) &= \exists y. R(x, y) \wedge ST(\phi) \\ ST(xa) &= a(x) \end{aligned} \quad (a \in \text{At})$$

and by commutation with all other constructs. In the converse direction, we translate $R(x, y)$ into $x\Diamond[z : z = y]$ and $a(x)$ into xa . In summary, *CPL over $\Lambda = \{\Diamond\} \cup \text{At}$ with the above semantics is expressively equivalent to the standard first-order correspondence language of modal logic.*

Graded Modal Logic. We obtain a variant of graded modal logic [Fin72] if we consider the similarity type $\Lambda = \{\langle k \rangle \mid k \geq 0\}$ where $\langle k \rangle$ reads as ‘more than k successors satisfy ...’. We interpret the ensuing logic over *multigraphs* [DV02], which are coalgebras for the *multiset functor* \mathcal{B} given on objects by

$$\mathcal{B}X = \{\mu : X \rightarrow \mathbb{N} \cup \{\infty\} \mid \mu \text{ a map}\}.$$

We use such a map $\mu : X \rightarrow \mathbb{N} \cup \{\infty\}$ like an integer-valued discrete measure on X , i.e. we write $\mu(A) = \sum_{x \in A} \mu(x)$ for $A \subseteq X$. Then, \mathcal{B} acts on maps $f : X \rightarrow Y$ by taking

image measures; i.e. $\mathcal{B}f(\mu)(B) = \mu(f^{-1}[B])$ for $B \subseteq Y$. We extend \mathcal{B} to a Λ -structure by stipulating

$$\llbracket \langle k \rangle \rrbracket_X(A) = \{\mu \in \mathcal{B}X \mid \mu(A) > k\}$$

to express that more than k successors (counted with multiplicities) have property A . Note that over Kripke frame, graded operators can be coded into standard first order logic; the difference with standard first-order logic arises through the multigraph semantics, for which the requisite expressive means arise only through the graded operators.

Continuing our line of routing examples, we can, given a \mathcal{B} -coalgebra $\gamma : C \rightarrow \mathcal{B}C$, think of $\gamma(c)(c')$ as the number of packets forwarded from c to c' in the past hour. In the presence of a binary relation $S(x, y)$ interpreted as above, the formula

$$\neg \exists y. (x \langle k \rangle [z : S(y, z)])$$

then expresses that there is no router y s.t. the total number of packets sent by x to nodes in y 's forwarding table in the past hour exceeds k .

Presburger modal logic and arithmetic. A more general set of operators than graded modal logic is that of positive Presburger modal logic [DL06], which admits integer linear inequalities $\sum_{i=1}^n a_i \cdot \#(\phi_i) > k$ among formulas where $a_i \geq 0$ for all i . We see such a formula as an application of an n -ary modality $L_k(a_1, \dots, a_n)$ to formulas ϕ_1, \dots, ϕ_n , and interpret this modality over the multiset functor \mathcal{B} as introduced above by the n -ary predicate lifting

$$\llbracket L_k(a_1, \dots, a_n) \rrbracket_X(A_1, \dots, A_n) = \{\mu \in \mathcal{B}X \mid \sum_{i=1}^n a_i \cdot \mu(A_i) > k\}.$$

In addition to the binary predicate S , let us also introduce unary predicate $O(x)$ expressing that x is an overloaded node. The formula

$$\forall x. (x(1 \cdot \#[y : S(x, y)] + 3 \cdot \#[y : O(y)] > 10,000) \rightarrow O(x))$$

means that, if the weighted number of packets sent by x to overloaded nodes combined with packets x sends to all nodes in its forwarding table exceeds 10,000, then x itself is overloaded.

Combination of Frame Classes. Frame classes can be combined: we can take $T = \mathcal{B} \times \mathcal{Q}\mathcal{Q}$ and combine operators for packet counting and subcommunity recognition. A formula

$$\neg x \Box [y : y \langle 30 \rangle [y : y \neq z]]$$

expresses then that the collection of those nodes which have forwarded more than 30 packages to servers different than z in the past hour is not a subcommunity recognized by x .

Probabilistic Modal Logic. The *discrete distribution functor* \mathcal{D} is defined on objects by

$$\mathcal{D}X = \{\mu : X \rightarrow [0, 1] \mid \sum_x \mu(x) = 1\},$$

and on morphisms by taking image measures exactly as for the multiset functor \mathcal{B} discussed earlier. Coalgebras for \mathcal{D} thus associate to every state a probability distribution over successor states; such structures are variously known as Markov chains, probabilistic transition systems, or type spaces. Taking the similarity type $\Lambda = \{\langle p \rangle \mid p \in [0, 1] \cap \mathbb{Q}\}$, with $\langle p \rangle$ read as ‘with probability more than p ’ (thus departing from the choice of operators L_p ‘with probability at least p ’ that we used in the introduction), we formally interpret $\langle p \rangle$ using the predicate lifting

$$\llbracket \langle p \rangle \rrbracket_X(A) = \{\mu \in \mathcal{D}X \mid \mu(A) \geq p\}$$

over \mathcal{D} . We thus obtain a form of probabilistic first-order logic for probabilistic transition systems that extends probabilistic modal logic [LS91, FH94, HM01]. Continuing our line of routing examples, if we interpret the transition probabilities as the likelihood of a server forwarding any given packet to another, then the formula

$$\forall x, y. (x \langle 1/2 \rangle [z : z = y] \rightarrow y : \langle 1/2 \rangle [z : z = x])$$

expresses a partial form of symmetric connectivity: whenever a server x prefers the connection to y in the sense that it will more likely than not route any given packet through y , then the same will hold in the other direction.

We obtain a version of this logic with finitely many modal operators in situations where all possible probabilities are contained in some finite set of rationals (such as when rolling a fair die). We then consider substructures of the form

$$\mathcal{D}_k X = \{\mu \in \mathcal{D}(X) \mid \mu(x) \in \{i/k \mid i = 0, \dots, k\}\},$$

restricting the modal operators to come from $\Lambda_k = \{\langle n/k \rangle \mid n = 0, \dots, k\}$.

Non-Monotonic Conditionals. An example of a binary modality is provided by (conditional) implication $>$, written in infix notation. Such operators are interpreted over a variety of semantic structures; one of these involves *selection function frames*, which in our terminology can be defined as coalgebras for the *selection function functor* \mathcal{S} . The latter acts on objects by

$$\mathcal{S}X = \{f : \mathcal{Q}X \rightarrow \mathcal{P}X\}$$

and, correspondingly, on maps $f : X \rightarrow Y$ by $\mathcal{S}f = \mathcal{Q}f \rightarrow \mathcal{P}f : \mathcal{S}X \rightarrow \mathcal{S}Y$, i.e. $\mathcal{S}f(g)(A) = f[g(f^{-1}[A])]$ for $A \subseteq X$ (recall that \mathcal{Q} denotes the contravariant powerset functor). We think of $f \in \mathcal{S}X$ as selecting the set $f(A)$ of ‘most typical’ worlds given a condition $A \subseteq X$. Over this functor, we interpret the conditional $>$ by the predicate lifting

$$\llbracket > \rrbracket_X(A, B) = \{f \in \mathcal{S}X \mid f(A) \cap B \neq \emptyset\}.$$

The formula $\phi > \psi$ expresses that ψ is typically possible under condition ϕ . This presentation of conditional logic is dual to the standard presentation [Che80] in terms of a binary operator \Rightarrow ‘if – then normally’, related to $>$ by $a > b \equiv \neg(a \Rightarrow \neg b)$. For our purposes, $>$ has the technical advantage of being bounded in the second argument in a sense that we will introduce in §3.

E.g. if we read the antecedent of \Rightarrow when applied at x as delineating a subcommunity in which x is currently active and the consequent as describing properties of those servers through which x will then normally route an incoming packet, then a formula of the form

$$\forall x, u. (\phi(u) \rightarrow x > [y : \phi(y)] [z : z = u])$$

says that if x is currently active in a subcommunity delineated by the formula $\phi(y)$ and u belongs to that subcommunity, then u is normally a possible target for packets forwarded by x .

3. COMPLETENESS

In §3.2 below, we propose an axiom system for CPL, sound wrt arbitrary structures (Theorem 3.13) and in §3.3 we show its completeness wrt structures s.t. each operator on every coordinate is either “sufficiently neighbourhood-like” or “sufficiently Kripke-like” (Theorem 3.15). As discussed in §3.5, even a mild relaxation of these conditions makes a generic completeness result impossible.

However, not only for the proof, but even for the statements of our completeness result, or of the axiomatization itself, we need some spadework.

3.1. S1SC and Boundedness. In order to state our axiomatization and completeness results, we need several notions from coalgebraic model theory. The first of them, central to the entire edifice, is that of *one-step satisfiability*.

Definition 3.1.

- Given a supply of primitive symbols D (which can be any set), define the set $\text{Prop}(D)$ of *boolean D -formulas* (or *propositions*) as

$$\mathbf{A}, \mathbf{B} ::= d \mid \mathbf{A} \rightarrow \mathbf{B} \mid \perp$$

where $d \in D$, and the set $\Lambda(D)$ of *modalized D -formulas* as

$$\Lambda(D) = \{\heartsuit d_1 \dots d_n \mid d_1, \dots, d_n \in D \text{ and } \heartsuit \in \Lambda \text{ is } n\text{-ary}\}.$$

Then the set $\text{Rank1}(D)$ of *rank-1 D -formulas* is defined as

$$\text{Rank1}(D) = \text{Prop}(\Lambda(\text{Prop}(D)));$$

in other words, a rank-1 formula is a Boolean combination of formulas consisting of a modality from Λ applied to Boolean combinations of atoms from D .

- Given a set C and a valuation $\tau : D \rightarrow \mathcal{P}(C)$, we extend τ to $\text{Prop}(D)$ using the Boolean algebra structure of $\mathcal{P}(C)$, and then write $C, \tau \models \mathbf{A}$ if $\tau(\mathbf{A}) = C$, for $\mathbf{A} \in \text{Prop}(D)$.
- Given the same data, we define the extension $\llbracket \phi \rrbracket_{TC, \tau} \subseteq TC$ of $\phi \in \text{Rank1}(D)$ by extending the assignment

$$\llbracket \heartsuit \mathbf{A}_1 \dots \mathbf{A}_n \rrbracket_{TX, \tau} = \llbracket \heartsuit \rrbracket_C(\tau(\mathbf{A}_1), \dots, \tau(\mathbf{A}_n))$$

using the Boolean algebra structure of $\mathcal{P}(TC)$.

- We then write $TC, \tau \models \phi$ if $\llbracket \phi \rrbracket_{TC, \tau} = TC$, and $t \models_{TC, \tau} \phi$ if $t \in \llbracket \phi \rrbracket_{TC, \tau}$.
- If $D \subseteq \mathcal{P}(C)$ and τ is just the inclusion, we will usually drop it from the notation; in particular, for subsets $Y_1, \dots, Y_n \subseteq C$ and $\heartsuit \in \Lambda$ n -ary, we write $t \models \heartsuit(Y_1, \dots, Y_n)$ to mean $t \in \llbracket \heartsuit \rrbracket_C(Y_1, \dots, Y_n)$.
- A set $\Xi \subseteq \text{Rank1}$ is *one-step satisfiable* w.r.t. τ if $\bigcap_{\phi \in \Xi} \llbracket \phi \rrbracket_{TC, \tau} \neq \emptyset$.

Just like in the case of coalgebraic modal logic (see §4 below), proof systems for CPL are best described in terms of rank-1 rules—or, more precisely, rule schemes, which describe the geometry of the Λ -structure under consideration. In our earlier papers [LPSS12, LPS13] and other references, these rules were described in two ultimately equivalent, but syntactically somewhat distinct ways. We give both definitions here:

Definition 3.2 (Hilbert-style one-step rules). Fix a collection \mathbf{sVar} of schematic variables $p, q, r \dots$

- A *Hilbert-style one-step rule* is of the form \mathbf{A}/\mathbf{P} , $\mathbf{A} \in \text{Prop}(\mathbf{sVar})$ and $\mathbf{P} \in \text{Rank1}(\mathbf{sVar})$.
- A one-step rule is a *one-step axiom scheme* if its premise is empty.
- A rule is *one-step sound* if

$$TC, \tau \models \mathbf{P} \text{ whenever } C, \tau \models \mathbf{A} \text{ for a valuation } \tau : \mathbf{sVar} \rightarrow \mathcal{P}(C).$$

- Given a set \mathcal{R} of basic one-step rules and a valuation $\tau : \mathbf{sVar} \rightarrow \mathcal{P}(C)$, a set $\Xi \subseteq \text{Rank1}(\mathbf{sVar})$ is *one-step consistent (with respect to τ)* [SP10c] if the set

$$\Xi \cup \{\mathbf{P}\sigma \mid \sigma : \mathbf{sVar} \rightarrow \text{Prop} \text{ and } \mathbf{A}/\mathbf{P} \text{ is a rule in } \mathcal{R} \text{ s.t. } C, \tau \models \mathbf{A}\sigma\}$$

is propositionally consistent.

From now on, we will only consider rule sets one-step sound relatively to a given Λ -structure, so the assumption of one-step soundness will not be mentioned explicitly.

Here is another approach to defining these rules:

Definition 3.3 (Gentzen-style one-step rules).

- A *Gentzen-style one-step rule* over a similarity type Λ is of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Gamma_R \Rightarrow \Delta_R}(R)$$

where

- $\Gamma_1, \dots, \Gamma_k, \Delta_1, \dots, \Delta_k$ are multisets of elements of \mathbf{sVar} ,
- Γ_R and Δ_R are multisets of elements of $\Lambda(\mathbf{sVar})$, i.e., $\Gamma_R \Rightarrow \Delta_R$ is of the form $\heartsuit_1 \vec{p}_1, \dots, \heartsuit_n \vec{p}_n \Rightarrow \heartsuit_{n+1} \vec{q}_1, \dots, \heartsuit_{n+m} \vec{q}_m$.
- For a Gentzen-style rule R , set

$$\text{Prem}(R) = \bigwedge \{ \bigwedge \Gamma_i \rightarrow \bigvee \Delta_i \mid 1 \leq i \leq k \} \in \text{Prop}(\mathbf{sVar})$$

$$\text{Conseq}(R) = \bigwedge \Gamma_R \rightarrow \bigvee \Delta_R \in \text{Prop}(\Lambda(\mathbf{sVar}))$$

The one-step Hilbert-style rule $\text{Prem}(R)/\text{Conseq}(R)$ is the (*Hilbert*) *flattening* of R .

- We say R is one-step sound if its flattening $\text{Prem}(R)/\text{Conseq}(R)$ is one-step sound as a Hilbert-style rule. The notion of one-step consistency relatively to a set \mathcal{R} of Gentzen-style one-step rules is analogously defined via its flattening.

These two definitions of one-step rules are mathematically equivalent, as discussed by Schröder [Sch07]. We will also need the main ingredient of the proof later on, e.g., for technicalities of completeness results, so let us recall it here:

Lemma 3.4. *For any Hilbert-style one-step rule \mathbf{A}/\mathbf{P} , there exists a substitution $\tau : \mathbf{sVar} \rightarrow \text{Prop}$ s.t. \mathbf{P} can be derived from \mathbf{A} and $\tau(\mathbf{P})$ using only boolean reasoning and the scheme*

$$((\mathbf{A}_1 \leftrightarrow \mathbf{B}_1) \wedge \dots \wedge (\mathbf{A}_n \leftrightarrow \mathbf{B}_n)) \rightarrow (\heartsuit \mathbf{A}_1 \dots \mathbf{A}_n \leftrightarrow \heartsuit \mathbf{B}_1 \dots \mathbf{B}_n).$$

Sketch. Assuming \mathbf{A} is a satisfiable boolean formula (otherwise the rule would be trivial), use the fact that as a satisfiable boolean term it has a *projective unifier* [Wro95, Ghi97] (see [BG11] for a recent overview): a substitution $\tau : \mathbf{sVar} \rightarrow \mathbf{Prop}$ s.t. both $\tau(\mathbf{A})$ and $\mathbf{A} \rightarrow (\mathbf{p} \leftrightarrow \tau(\mathbf{p}))$ (for any $\mathbf{p} \in \mathbf{sVar}$) are boolean tautologies. \square

Definition 3.5. A rule set \mathcal{R} is *strongly 1-step complete (S1SC)* for a Λ -structure if for every $C \in \mathbf{Set}$, any $\Xi \subseteq \mathbf{Rank1}(\mathbf{sVar})$ and any $\tau : \mathbf{sVar} \rightarrow \mathcal{P}(C)$, Ξ is one-step satisfiable wrt τ whenever it is one-step consistent wrt τ .

Remark 3.6. As noted in the original reference [SP10c, Remark 55], we can give a more abstract statement of S1SC, recognizable also to readers familiar with more categorical presentations of coalgebraic modal logic, such as Kurz and Rosicky [KR12]. Every signature Λ together with a given set of one-step axiom schemes (equivalently, making the corresponding set of one-step rules sound) can be encoded disregarding concrete syntax by its *functorial presentation* [KKP04] (cf. also [SP10c, Definition 28]) as an endofunctor L_Λ on the category of boolean algebras \mathbf{BA} . \mathbf{BA} is dually adjoint to \mathbf{Set} , with the adjunction given by the contravariant powerset functor² $\overline{\mathcal{Q}}$ and the functor \mathcal{S} taking a Boolean algebra to the set of its ultrafilters:

$$L_\Lambda \hookrightarrow \mathbf{BA} \xrightleftharpoons[\mathcal{S}]{\overline{\mathcal{Q}}} \mathbf{Set} \hookleftarrow T \quad (3.1)$$

The information contained in each Λ -structure can be then more abstractly encoded by $\delta : L_\Lambda \overline{\mathcal{Q}} \rightarrow \overline{\mathcal{Q}} T$ [KKP04] and the *canonical* (neighbourhood-like) structure for Λ is given by $M_\Lambda = \mathcal{S} L_\Lambda \overline{\mathcal{Q}}$. For every Λ -structure, we can define a canonical *structure morphism* [SP10c, p. 1121] to M_Λ by composing the counit of the above adjunction with $\mathcal{S}\delta$, and S1SC effectively requires that this structure morphism is surjective. As the next remark indicates, having such a surjective morphism onto the canonical neighbourhood-like structure is a restrictive condition.

Remark 3.7. Coalition logic [Pau02] and, essentially equivalently, the next-step fragment of alternating-time temporal logic [AHK02], have modalities $[Q]$ indexed over *coalitions* Q , which are subsets of a fixed finite set N of *agents*; the operator $[Q]$ reads ‘the coalition Q of players can enforce ... in the next step’. The semantics is formulated over structures called *game frames* or *concurrent game structures*, i.e., coalgebras for the functor

$$\mathcal{G}X = \{((S_i)_{i \in P}, f : (\prod_{i \in N} S_i) \rightarrow X) \mid \emptyset \neq S_i \subseteq N\}$$

where S_i is thought of as the set of moves available to agent $i \in N$ and f is an *outcome function* that determines the next state of the game, depending on the moves chosen by the agents (we restrict to finitely many moves per agent as in alternating-time temporal logic). For notational convenience, given a coalition $Q = \{q_1, \dots, q_k\} \subseteq N$ and moves $s_{q_1} \in S_{q_1}, \dots, s_{q_k} \in S_{q_k}$, we write $s_Q = (s_q)_{q \in Q}$ and $S_Q = S_{q_1} \times \dots \times S_{q_k}$ (so that $s_Q \in S_Q$). Given $s_Q \in S_Q$ and $s_{N \setminus Q} \in S_{N \setminus Q}$, we write $(s_Q, s_{N \setminus Q})$ for the evident induced element of S_N .

An alternative semantics of the coalitional operators is provided by *effectivity functions*. These are functions E assigning to each coalition Q a set $E(Q)$ of properties that Q can enforce. Explicitly, a concurrent game $G = ((S_i)_{i \in P}, f) \in \mathcal{G}X$ induces an effectivity function E_G by

$$E_G(Q) = \{A \subseteq X \mid \exists s_Q \in S_Q. \forall s_{N \setminus Q} \in S_{N \setminus Q}. f(s_Q, s_{N \setminus Q}) \in A\}.$$

²We write here $\overline{\mathcal{Q}}$ to stress that we change the target category.

Effectivity functions congregate into a functor \mathcal{E} , a subfunctor of a product of neighbourhood functors. The modal operators $[Q]$ are interpreted over effectivity functions in the usual style of neighbourhood semantics, i.e. by

$$\llbracket [Q] \rrbracket_X(A) = \{E \in \mathcal{E}X \mid A \in E(Q)\}.$$

Composing this semantics with the above-defined projection from concurrent games to effectivity functions yields the interpretation of the coalitional modalities $[Q]$ over \mathcal{G} ; this reproduces the standard semantics of coalition logic and alternating time temporal logic.

Now Theorem 3.2 in [Pau02] states that an effectivity function $E \in \mathcal{E}(X)$ is of the form E_G for some G iff it is *playable*, i.e. satisfies the following properties:

- For all Q , $\emptyset \notin E(Q) \ni X$
- E is *outcome-monotonic*, i.e. each $E(Q)$ is upwards closed under set inclusion.
- E is *N-maximal*, i.e. for all $A \subseteq X$, either $X \setminus A \in E(\emptyset)$ or $A \in E(N)$.
- E is *superadditive*, i.e. whenever $A_1 \in E(Q_1)$ and $A_2 \in E(Q_2)$ for disjoint coalitions Q_1, Q_2 , then $A_1 \cap A_2 \in E(Q_1 \cup Q_2)$.

If this were the case, then coalition logic interpreted over either concurrent games or playable effectivity functions would be S1SC, as the above conditions on playable effectivity functions amount to the satisfaction of finitary one-step axioms (specifically, $\neg[Q]\perp$, $[Q]\top$, $[Q](a \wedge b) \rightarrow [Q]a$, $[\emptyset]\neg a \vee [N]a$, and $[Q_1]a \wedge [Q_2]b \rightarrow [Q_1 \cup Q_2](a \wedge b)$ for $Q_1 \cap Q_2 = \emptyset$) [SP10c], and we claimed as much in the conference version [LPSS12]. However, it turns out that Theorem 3.2 in [Pau02] is not in fact entirely correct, and once fixed no longer implies that coalition logic is S1SC. To see this, note that for every effectivity function of the form E_G , $E_G(\emptyset)$ must have a least element, equivalently be closed under intersections: every element of $E_G(\emptyset)$ must contain the set

$$A = \{f(s_N) \mid s_N \in S_N\},$$

and this set is itself in $E_G(\emptyset)$. This condition is however not satisfied by all playable effectivity functions in the above sense: take X to be some infinite set, pick a non-fixed ultrafilter U on X , and put $E(Q) = U$ for all coalitions Q . This defines a playable effectivity function but $E(\emptyset) = U$ has no least element. Adding the condition that $E(\emptyset)$ has a least element to the definition of playability does fix the theorem, but this condition is not expressible by a finitary one-step axiom and hence we do not obtain S1SC for coalition logic as a corollary.

As indicated above, we have alternative conditions that ensure completeness [SP10b]:

Definition 3.8.

- A modal operator \heartsuit is *k-bounded* in the i -th argument for $k \in \mathbb{N}$ and with respect to a Λ -structure T if for every $C \in \text{Set}$ and every $\overline{A} \subseteq C$,

$$\llbracket \heartsuit \rrbracket_C(A_1, \dots, A_n) = \bigcup_{B \subseteq A_i, \#B \leq k} \llbracket \heartsuit \rrbracket_C(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n).$$

This implies in particular that \heartsuit is monotonic in the i -th argument.

- A *boundedness signature* for Λ is a function $\flat\Lambda$ assigning to every $\heartsuit \in \Lambda$ a vector of elements of $\mathbb{N} \cup \{\infty\}$ of length $\text{ar}\heartsuit$, i.e. an element of $(\mathbb{N} \cup \{\infty\})^{\text{ar}\heartsuit}$.
- Being *∞ -bounded* is a condition trivially satisfied by all operators, i.e., every operator is “ ∞ -bounded” in each coordinate.
- We say that $\flat\Lambda$ is *adequate* for a Λ -structure over T if every modal operator $\heartsuit \in \Lambda$ is $\flat\Lambda(\heartsuit)(i)$ -bounded in i every $i \leq \text{ar}(\heartsuit)$.

- We say that Λ is $\flat\Lambda$ -*bounded* w.r.t T if $\flat\Lambda$ is adequate for the structure in question and the codomain of $\flat\Lambda$ does not contain ∞ .
- We say that Λ is *bounded* w.r.t. T if it is $\flat\Lambda$ -*bounded* w.r.t. T for *some* $\flat\Lambda$. That is, every modal operator $\heartsuit \in \Lambda$ for every i smaller than its arity is $k_{\heartsuit,i}$ -bounded in i for some $k_{\heartsuit,i}$.

Example 3.9. Here are some examples of boundedness signatures adequate for structures under consideration:

- for the neighbourhood case, $\flat\Lambda(\Box) = (\infty)$,
- for the Kripke case, $\flat\Lambda(\Diamond) = (1)$,
- for graded modalities, $\flat\Lambda(\langle k \rangle) = (k + 1)$,
- for positive Presburger logic,

$$\flat\Lambda(L_k(a_1, \dots, a_n)) = ((k + 1) \operatorname{div} a_1 + 1, \dots, (k + 1) \operatorname{div} a_n + 1),$$

- for the discrete distribution functor \mathcal{D} , $\flat\Lambda(\langle p \rangle) = (\infty)$,
- for its finite variant \mathcal{D}_k , $\flat\Lambda(\langle n/k \rangle) = (n)$,
- for non-monotonic conditionals, $\flat\Lambda(>) = (\infty, 1)$.

Note that, e.g., the neighbourhood modality clearly fails to be bounded; boundedness is a “Kripke-like” property. It allows us to broaden the scope of our completeness results to setups where full S1SC would be too much to ask, i.e., to leave the neighbourhood-like setting. This is done by requiring S1SC on suitable coordinates only for valuations of schematic variables in *finite* sets. In order to make this precise so that we can cover mixed cases, such as those of non-monotonic conditionals, some care is needed.

Definition 3.10.

- The *colouring* function $\flat : \mathbb{N} \cup \{\infty\} \rightarrow \{fin, \infty\}$ assigns *fin* to elements of \mathbb{N} and $\flat(\infty) = \infty$. It is extended pointwise to $(\mathbb{N} \cup \{\infty\})^{\operatorname{ar}\heartsuit}$ and $\flat\Lambda$ is defined as the composition of $\flat\Lambda$ with this pointwise extension.
- Let $\mathbf{c} : \mathbf{sVar} \rightarrow \{fin, \infty\}$ be a colouring of the set of schematic variables. Define the set of $\flat\Lambda, \mathbf{c}$ -*coloured modalities* as

$$\flat\Lambda_{\mathbf{c}} = \{\heartsuit \mathbf{p}_1 \dots \mathbf{p}_{\operatorname{ar}\Lambda} \mid \mathbf{p}_1 \dots \mathbf{p}_{\operatorname{ar}\Lambda} \in \mathbf{sVar} \text{ and } (\mathbf{c}\mathbf{p}_1, \dots, \mathbf{c}\mathbf{p}_{\operatorname{ar}\Lambda}) = \flat\Lambda(\heartsuit)\}.$$

- A valuation $\tau : \mathbf{sVar} \rightarrow \mathcal{P}(C)$ respects \mathbf{c} iff $\tau(\mathbf{p}_i) \in \mathcal{P}_{\mathbf{c}\mathbf{p}_i}$, where we recall that \mathcal{P}_{fin} is finite powerset, and \mathcal{P}_{∞} is simply \mathcal{P} .
- A Gentzen-style one-step rule R is $\flat\Lambda, \mathbf{c}$ -*compatible* if it is of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\Gamma_R \Rightarrow \Delta_R}(R)$$

where

- $\Gamma_1, \dots, \Gamma_k, \Delta_1, \dots, \Delta_k$ are multisets of elements of \mathbf{sVar} ,
- Γ_R and Δ_R are multisets of elements of $\flat\Lambda_{\mathbf{c}}$.

- For a Gentzen-style rule, write $\flat\Lambda_{\mathbf{c}}(R)$ for the set of $\flat\Lambda, \mathbf{c}$ -compatible variants of R obtained by renaming of schematic variables. For a Hilbert-style rule, $\flat\Lambda_{\mathbf{c}}(R)$ is obtained via its Gentzen-style counterpart produced in Lemma 3.4. Finally, $\flat\Lambda_{\mathbf{c}}(\mathcal{R}) = \{\flat\Lambda_{\mathbf{c}}(R) \mid R \in \mathcal{R}\}$.
- A set $\Xi \subseteq \operatorname{Rank1}(\mathbf{sVar})$ is \mathbf{c} -*consistent* wrt τ if its sum with the set

$$\{\mathbf{P}\sigma \mid \sigma : \mathbf{sVar} \rightarrow \operatorname{Prop} \text{ and } \mathbf{A}/\mathbf{P} \in \flat\Lambda_{\mathbf{c}}(\mathcal{R}) \text{ s.t. } C, \tau \models \mathbf{A}\sigma\}$$

is propositionally consistent.

- We say that a set of rules \mathcal{R} is $\mathfrak{b}\Lambda$ -S1SC if
 for every $C \in \text{Set}$, any $\Xi \subseteq \text{Rank1}(\text{sVar})$, any colouring \mathfrak{c} and any τ respecting \mathfrak{c} ,
 Ξ is one-step satisfiable wrt τ whenever it is \mathfrak{c} -consistent wrt τ .

In the case of a bounded Λ , i.e., when $\mathfrak{b}\Lambda$ does not contain ∞ in its range (for any \heartsuit and any coordinate), rules are $\mathfrak{b}\Lambda$, \mathfrak{c} -compatible with those \mathfrak{c} which colour all schematic variables with *fin*. Valuations respecting such \mathfrak{c} are precisely those which interpret schematic variables as finite subsets of C . Thus, another way to state the above definition for a bounded Λ would be as the variant of Definition 3.5 obtained by replacing \mathcal{P} with \mathcal{P}_{fin} . That is, we have:

Fact 3.11. In the case of a bounded Λ , i.e., when $\mathfrak{b}\Lambda$ does not contain ∞ in its range (for any \heartsuit and any coordinate), a rule set \mathcal{R} is $\mathfrak{b}\Lambda$ -S1SC iff

- for every $C \in \text{Set}$, any $\Xi \subseteq \text{Rank1}(\text{sVar})$ and any $\tau : \text{sVar} \rightarrow \mathcal{P}_{fin}(C)$,
 Ξ is one-step satisfiable wrt τ whenever it is one-step consistent wrt τ .

In this case, we can use the name *finitary S1SC* and the case of the modal signature $\{>\}$ of non-monotonic conditionals with $\mathfrak{b}\Lambda(>) = (\infty, 1)$ leads to the condition which can be called (*S1SC*, *finitary S1SC*) [SP10b].

3.2. Our Axiomatization. We are finally ready to present our axioms for CPL in Table 1. Axioms EN1–EN6 are just those of Enderton, with EN6.2 an additional clause to cover the case of modal formulas. The α -renaming axiom ALPHA is needed because our syntax features separate comprehension variables. The CONG axioms is the basic one for Chang’s formalism, and in fact all that is needed in neighbourhood semantics; for most standard sets of one-step rules (including, indeed, those for the neighbourhood semantics itself), however, it is redundant, as discussed in Remark 3.12 below. This is due to the ONESTEP(\mathcal{R}) axiom scheme incorporating the entire propositional framework of coalgebraic logic. Finally, BDPL $_{\mathfrak{b}\Lambda}$ is an axiom for operators bounded in suitable coordinates. It is important to notice that boundedness is not expressible as a *sentence* or *formula* in weak frameworks; in languages like $H_\Lambda(@)$, it can only be expressed by a non-standard rule [SP10b].

Remark 3.12. In fact, for most natural sets \mathcal{R} of one-step rules, the corresponding instances of ONESTEP(\mathcal{R}) actually make CONG redundant. This follows from earlier results on *one-step cut-free completeness* [PS10, Proposition 5.12]. We are going to devote more space to the issue in the cut-elimination section, see in particular Corollary 6.8 below.

Let $\Gamma, \Delta \subseteq \text{CPL}(\Lambda, \Sigma)$, let \mathcal{R} be a set of one-step rules and $\phi \in \text{CPL}(\Lambda, \Sigma)$. Write $\Gamma \vdash_{\mathfrak{b}\Lambda}^{\mathcal{H}\mathcal{R}} \phi$ if there are $\gamma_1, \dots, \gamma_n \in \Gamma$ s.t. $\gamma_1 \rightarrow \dots \rightarrow \gamma_n \rightarrow \phi$ can be deduced from EN1–EN6, ALPHA, CONG, ONESTEP(\mathcal{R}) and BDPL $_{\mathfrak{b}\Lambda}$ in Table 1 using **only Modus Ponens**. This clearly defines a *finitary deducibility relation* in the sense of Goldblatt [Gol93, Sec. 8.1] and being $\vdash_{\mathfrak{b}\Lambda}^{\mathcal{H}\mathcal{R}}$ -consistent is equivalent with being *finitely* $\vdash_{\mathfrak{b}\Lambda}^{\mathcal{H}\mathcal{R}}$ -consistent in his sense, that is, $\Gamma \vdash_{\mathfrak{b}\Lambda}^{\mathcal{H}\mathcal{R}} \perp$ iff there is $\Gamma_0 \subseteq_{fin} \Gamma$ s.t. $\Gamma_0 \vdash_{\mathfrak{b}\Lambda}^{\mathcal{H}\mathcal{R}} \perp$. Note that the axiom CONG is in fact (a syntactic variant of) an axiom already introduced by Chang [Cha73].

Theorem 3.13 (Soundness). *Whenever a Λ -structure over T is adequate for $\mathfrak{b}\Lambda$, all the axioms in Table 1 hold in every coalgebraic Λ -model and the set of formulas valid in such a model is closed under $\vdash_{\mathfrak{b}\Lambda}^{\mathcal{H}\mathcal{R}}$.*

Table 1: Hilbert-style Calculus \mathcal{HR}

The axioms are modelled after those of Enderton [End01].

Everywhere below, $\forall \vec{y}.$ denotes a sequence of universal quantifiers of arbitrary length, possibly empty.

EN1: all propositional tautologies. Can be axiomatized, e.g., by:

- $\forall \vec{y}. (\phi \rightarrow (\psi \rightarrow \phi))$
- $\forall \vec{y}. ((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$
- $\forall \vec{y}. (\perp \rightarrow \phi)$
- $\forall \vec{y}. (((\phi \rightarrow \perp) \rightarrow \perp) \rightarrow \phi)$

For EN2 and ALPHA below, recall that whenever we write a substitution we implicitly impose the assumption that the substituted term is actually substitutable.

EN2: $\forall \vec{y}. (\forall x. \phi \rightarrow \phi[z/x])$

EN3: $\forall \vec{y}. (\forall x. (\phi \rightarrow \psi) \rightarrow (\forall x. \phi \rightarrow \forall x. \psi))$

EN4: $\forall \vec{y}. (\phi \rightarrow \forall x. \phi)$ if x is fresh for ϕ

EN5: $\forall \vec{y}. (x = x)$

EN6.1: $\forall \vec{y}. (x = z \rightarrow P(\vec{u}, x, \vec{v}) \rightarrow P(\vec{u}, z, \vec{v}))$ for $P \in \Sigma \cup \{=\}$

EN6.2: $\forall \vec{y}. (x = z \rightarrow x \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n] \rightarrow z \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n])$

ALPHA: $\forall \vec{y}. ((x \heartsuit \dots [z : \phi] \dots) \rightarrow (x \heartsuit \dots [u : \phi[u/z]] \dots))$

CONG: $\forall \vec{y}. (\forall x. (\bigwedge_{i=1}^n (\phi_i \leftrightarrow \psi_i)) \rightarrow \forall x. ((x \heartsuit [x : \phi_1] \dots [x : \phi_n]) \leftrightarrow (x \heartsuit [x : \psi_1] \dots [x : \psi_n])))$
(redundant for *one-step cut-free complete* rule sets, see Remark 3.12)

ONESTEP(\mathcal{R}): $\forall \vec{y}. \forall z. (\forall x. \sigma(\text{Prem}(R)) \rightarrow [\sigma, x, z] \text{Conseq}(R))$ where

- R ranges over the one-step rules in \mathcal{R}
- σ sends each \mathbf{p}_i to a formula of \mathcal{L} and $[\sigma, x, z]$ is the inductive extension of the map sending each $\heartsuit_i \vec{\mathbf{p}}_i$ to $z \heartsuit_i [x : \sigma(\mathbf{p}_i^1)] \dots [x : \sigma(\mathbf{p}_i^{a(i)})]$
- $\text{Prem}(R) = \bigwedge \{ \bigwedge \Gamma_i \rightarrow \bigvee \Delta_i \mid 1 \leq i \leq k \}$ and $\text{Conseq}(R) = \bigwedge \Gamma_R \rightarrow \bigvee \Delta_R$ represent the premises and conclusion of the rule as in Definition 3.3.

BDPL $_{\mathbf{b}\Lambda}$: An additional axiom scheme when $\mathbf{b}\Lambda(\heartsuit)(i) \neq \infty$ and \vec{z} fresh for $y_i, \vec{\phi}$

$$\forall \vec{y}. (x \heartsuit \dots [y_i : \phi_i] \dots \leftrightarrow \exists z_1 \dots z_{\mathbf{b}\Lambda(i)}. (\bigwedge_{j \leq \mathbf{b}\Lambda(i)} \phi_i[z_j/y_i] \wedge x \heartsuit \dots [y_i : \bigvee_{j \leq \mathbf{b}\Lambda(i)} y_i = z_j] \dots))$$

Definition 3.14. For any Λ , \mathcal{R} and $\mathbf{b}\Lambda$, we say that the inference system given by $\vdash_{\mathbf{b}\Lambda}^{\mathcal{HR}}$ is *strongly complete* for a given Λ -structure based on T if for any set of sentences $\Gamma \in \text{CPL}(\Lambda, \Sigma)$, $\Gamma \vdash_{\mathbf{b}\Lambda}^{\mathcal{HR}} \perp$ holds **if and only if** there is a coalgebraic Λ -model for Γ .

Theorem 3.15 (Strong Completeness). *Whenever a set of rules \mathcal{R} is $\mathbf{b}\Lambda$ -S1SC for a Λ -structure over T that is adequate for $\mathbf{b}\Lambda$, then $\vdash_{\mathbf{b}\Lambda}^{\mathcal{HR}}$ is strongly complete for this structure.*

Example 3.16. For the examples presented in §2, the situation is as follows. Completeness holds for neighbourhood models as they have a strongly one-step complete axiomatisation. For all others, but excluding non-monotonic conditionals, finitary one-step complete axiomatisations exist. As discussed above (cf. Example 3.9), boundedness holds for relational models, graded modal logic and the logic of finite probabilities (interpreted over \mathcal{D}_k -coalgebras) whereas conditional logic is covered as a mixed case.

3.3. Proof of The Completeness Theorem. First, we introduce machinery proposed in [Gol93]. Consider any $Fr \subseteq \text{CPL}(\Lambda, \Sigma)$ closed under propositional connectives. Fr can be, for example, the set of all formulas whose free variables are contained in a fixed subset of iVar , the set of all sentences and the entire $\text{CPL}(\Lambda, \Sigma)$ itself being the two borderline cases. Any set $Inf \subseteq \mathcal{P}(Fr) \times Fr$ will be called, following Goldblatt, *a set of inferences*. For any $inf = (\Pi, \chi) \in Inf$ and any $\Gamma \subseteq Fr$, we say that

- Γ *respects* inf if $\Gamma \vdash_{b\Lambda}^{\mathcal{HR}} \chi$ whenever $\Gamma \vdash_{b\Lambda}^{\mathcal{HR}} \phi$ for all $\phi \in \Pi$,
- Γ *is closed under* inf if $\chi \in \Gamma$ whenever $\Pi \subseteq \Gamma$,
- Γ *respects* Inf iff it respects each member of Inf ,
- Γ *is closed under* Inf iff it is closed under each member of Inf .

Theorem 3.17 (Goldblatt’s Abstract Henkin Principle [Gol93]). *If Inf is a set of inferences in Fr of an infinite cardinality κ and Γ is a $\vdash_{b\Lambda}^{\mathcal{HR}}$ -consistent subset of Fr satisfying in addition:*

$$\forall X \subseteq Fr. |X| < \kappa \text{ implies that } \Gamma \cup X \text{ respects } Inf \quad (3.2)$$

(i.e., every κ -finite extension of Γ respects Inf), then Γ has a maximally $\vdash_{b\Lambda}^{\mathcal{HR}}$ -consistent extension in Fr which is closed under Inf .

Remark 3.18. We emphasize that speaking about *inferences* in Goldblatt’s sense being infinite sets does *not* mean that *deductions* in the axiom system for CPL use infinitary rules. As stated above, the only inference rule in our system is ordinary Modus Ponens. Even the one-step rules defined above (which are not infinitary anyway) can be written as sentence schemes ONESTEP thanks to the use of quantifiers.

We further point out that an Enderton-style axiomatization does not involve the generalization rule: if x is a free variable in ϕ , it is not necessarily the case that $\phi \vdash_{b\Lambda}^{\mathcal{HR}} \forall x.\phi$ (this is not in contradiction to completeness: the rule is sound in the sense that validity of the premise implies validity of the conclusion, but its conclusion is not a logical consequence of its premise). This makes it enjoy a rather rare property for an axiomatization of FOL: a deduction theorem in exactly the same form as propositional logic, i.e., $\Gamma \cup \{\phi\} \vdash_{b\Lambda}^{\mathcal{HR}} \psi$ iff $\Gamma \vdash_{b\Lambda}^{\mathcal{HR}} \phi \rightarrow \psi$ (cf. [End01, p. 118]). This will also allow us to give our Henkin-style proofs *without introducing additional constants*—the role of Henkin constants for existentially quantified variables will be played by the variables themselves. The only disadvantage of this approach would be that if we consider uncountable Λ or Σ , we would also need to allow uncountably many elements of iVar , something we highlight in the statement of several lemmas and claims below.

Let us recall the crucial ingredient in Henkin-style completeness proofs: the notion of quasi-Henkin model and its associated Truth Lemma. This is inspired by previously announced completeness proofs for coalgebraic hybrid logic [SP10b]; we discuss the relationship in detail in Remark 3.24 and §4 below.

Definition 3.19. Let Γ be a maximal consistent set (MCS) of formulas. Define $C_\Gamma = \{|x| : x \text{ is a variable}\}$, where $|x| = \{z : x = z \in \Gamma\}$, and put $I_\Gamma(P) = \{(|x_1|, \dots, |x_n|) : P(x_1, \dots, x_n) \in \Gamma\}$. Set $\hat{\phi}^{y_i} = \{|z| : \phi[z/y_i] \in \Gamma\}$, to be thought of as *the set of variables satisfying ϕ according to Γ* (when y_i is taken to be the *argument variable* or the *context hole*). We say that $(C_\Gamma, \gamma, I_\Gamma)$ is a *quasi-Henkin coalgebraic model* if, for any variables x, y_1, \dots, y_n and any formulas $\psi, \phi_1, \dots, \phi_n$,

$$\exists x.\psi \in \Gamma \implies \text{for some } y_i, y_i \in \hat{\psi}^x. \quad (3.3)$$

(note that the converse implication holds for any MCS) and

$$x\heartsuit[y_1 : \phi_1] \cdots [y_n : \phi_n] \in \Gamma \iff \gamma(|x|) \in \llbracket \heartsuit \rrbracket_{C_\Gamma}(\widehat{\phi_1}^{y_1}, \dots, \widehat{\phi_n}^{y_n}). \quad (3.4)$$

In a quasi-Henkin model, define the *canonical variable assignment* v_Γ by $v_\Gamma(x) = |x|$.

Lemma 3.20 (Truth Lemma). *Let Γ be a maximal consistent set of formulas and $\mathfrak{M}_\Gamma = (C_\Gamma, \gamma, I_\Gamma)$ a quasi-Henkin coalgebraic model. Then, for every formula ϕ ,*

$$\mathfrak{M}_\Gamma, v_\Gamma \models \phi \iff \phi \in \Gamma. \quad (3.5)$$

Proof. By induction on ϕ . An auxiliary fact we need is that whenever ϕ satisfies the inductive claim (3.5), then $\llbracket \phi \rrbracket^y = \widehat{\phi}^y$ (recall the left-hand side is a piece of notation introduced when defining the notion of satisfaction), which can be shown in the following way. Let $|z| \in C_\Gamma$. Then we have

$$\begin{aligned} \mathfrak{M}_\Gamma, v_\Gamma[|z|/y] \models \phi &\iff \mathfrak{M}_\Gamma, v_\Gamma \models \phi[z/y] \\ &\iff \phi[z/y] \in \Gamma \quad \text{by (3.5),} \end{aligned}$$

as desired.

The base case of induction for atomic formulas follows now from the definitions of C_Γ and I_Γ , the Boolean cases from the fact that we are dealing with a MCS, and the case for quantifiers directly from Condition 3.3. For the modal case, where $\phi \equiv x\heartsuit[y : \psi]$, we have:

$$\begin{aligned} \mathfrak{M}_\Gamma, v_\Gamma \models x\heartsuit[y : \psi] &\iff \gamma(v_\Gamma(x)) \in \llbracket \heartsuit \rrbracket_{C_\Gamma}(\llbracket \psi \rrbracket^y) \quad \text{by definition} \\ &\iff \gamma(|x|) \in \llbracket \heartsuit \rrbracket_{C_\Gamma}(\widehat{\psi}^y) \quad \text{by the auxiliary fact above} \\ &\iff x\heartsuit[y : \psi] \in \Gamma \quad \text{by the quasi-Henkin property.} \end{aligned}$$

□

Next, we need to find a suitable candidate for an MCS from which to build our quasi-Henkin model. Consider the following sets of inferences:

$$\begin{aligned} Inf_{NAMEa} &= \{ \langle \{ \phi[z/x] \mid z \in \text{iVar} \}, \forall x. \phi \rangle \mid \phi \in \text{CPL}(\Lambda, \Sigma), x \in \text{iVar} \} \\ Inf_{NAMEb} &= \{ \langle \{ (\phi_1 \leftrightarrow \psi_1)[z/x], \dots, (\phi_n \leftrightarrow \psi_n)[z/x] \mid z \in \text{iVar} \}, \\ &\quad \forall x. (x\heartsuit[x : \phi_1] \dots [x : \phi_n] \leftrightarrow x\heartsuit[x : \psi_1] \dots [x : \psi_n]) \rangle \mid \overline{\phi}, \overline{\psi} \in \text{CPL}(\Lambda, \Sigma), x \in \text{iVar} \} \\ Inf_{NAME} &= Inf_{NAMEa} \cup Inf_{NAMEb} \\ Inf_{b\Lambda} &= \{ \langle \{ \bigwedge_{j \leq b\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \rightarrow \neg x\heartsuit \dots [y_i : \bigvee_{j \leq b\Lambda(\heartsuit)(i)} y_i = z_j] \dots \mid \overline{z} \in \text{iVar} \}, \\ &\quad \neg x\heartsuit \dots [y_i : \phi_i] \dots \rangle \mid \overline{\phi} \in \text{CPL}(\Lambda, \Sigma), x \in \text{iVar}, \heartsuit \in \Lambda, b\Lambda(\heartsuit)(i) \neq \infty \} \\ Inf &= Inf_{NAME} \cup Inf_{b\Lambda} \end{aligned}$$

Let us begin with

Claim 3.21. *Assume $|\text{iVar}| = \kappa \geq |\Lambda \cup \Sigma|$. Then any $\vdash_{b\Lambda}^{\mathcal{HR}}$ -consistent set of formulas Γ s.t. $|\{x \in \text{iVar} \mid x \text{ fresh for } \Gamma\}| = \kappa$ (in particular, any consistent set of sentences) satisfies condition 3.2 of Theorem 3.17 for Inf_{NAME} .*

Proof. We begin by observing that

$$(a) \text{ If } \Gamma' \vdash_{b\Lambda}^{\mathcal{HR}} \phi[z/x] \text{ and } z \text{ is fresh for } \Gamma', x, \phi, \text{ then } \Gamma' \vdash_{b\Lambda}^{\mathcal{HR}} \forall x. \phi$$

The proof of this fact is perfectly standard, but working with an Enderton-style axiomatization is particularly convenient for such reasoning: We have a finite $\Gamma'_0 \subseteq_{fin} \Gamma'$ s.t. $\Gamma'_0 \vdash_{b\Lambda}^{\mathcal{HR}} \phi[z/x]$. Then one uses the Deduction Theorem (cf. Remark 3.18) to obtain $\vdash_{b\Lambda}^{\mathcal{HR}} \bigwedge \Gamma'_0 \rightarrow \phi[z/x]$. However, even with an Enderton-style axiomatization it is still the case³ that $\vdash_{b\Lambda}^{\mathcal{HR}} \chi$ implies $\vdash_{b\Lambda}^{\mathcal{HR}} \forall z.\chi$, hence $\vdash_{b\Lambda}^{\mathcal{HR}} \forall z.(\bigwedge \Gamma'_0 \rightarrow \phi[z/x])$. The rest is an easy exercise using EN3, EN4 and renaming of bound variables thanks to EN2.

The condition (a) tells us that Γ itself does respect Inf_{NAMEa} by assumption. But if $|X| < \kappa$, then there are κ -many $z \in \mathbf{iVar}$ that are fresh for $\Gamma \cup X \cup \{\phi\}$. For any such z , (a) would hold also for $\Gamma' = \Gamma \cup X$. This gives condition 3.2 for Inf_{NAMEa} .

For Inf_{NAMEb} , let us observe that (a) allows to infer that

$$\begin{aligned} &\text{If } \Gamma' \vdash_{b\Lambda}^{\mathcal{HR}} (\phi_1 \leftrightarrow \psi_1)[z/x] \wedge \dots \wedge (\phi_n \leftrightarrow \psi_n)[z/x] \text{ and } z \text{ fresh for } \Gamma', \bar{\phi}, \bar{\psi}, x, \text{ then} \\ &\quad \Gamma \vdash_{b\Lambda}^{\mathcal{HR}} \forall x. ((\phi_1 \leftrightarrow \psi_1) \wedge \dots \wedge (\phi_n \leftrightarrow \psi_n)). \end{aligned}$$

Now an application of CONG completes the proof of the claim. \square

Claim 3.22. *Assume $|\mathbf{iVar}| = \kappa \geq |\Lambda \cup \Sigma \cup \omega|$. Then any $\vdash_{b\Lambda}^{\mathcal{HR}}$ -consistent set of formulas Γ s.t. $|\{x \in \mathbf{iVar} \mid x \text{ fresh for } \Gamma\}| = \kappa$ (in particular, a consistent set of sentences) satisfies condition 3.2 of Theorem 3.17 for $Inf_{b\Lambda}$.*

Proof. We begin by observing that

$$\begin{aligned} &\text{(b) If } \Gamma' \vdash_{b\Lambda}^{\mathcal{HR}} \neg(\bigwedge_{j \leq b\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \wedge x \heartsuit \dots [y_i : \bigvee_{j \leq b\Lambda(\heartsuit)(i)} y_i = z_j] \dots) \text{ for some} \\ &\quad \bar{z} \text{ fresh for } \Gamma', x, y_i, \bar{\phi}, \text{ then } \Gamma' \vdash_{b\Lambda}^{\mathcal{HR}} \neg x \heartsuit \dots [y_i : \phi_i] \dots \end{aligned}$$

This is shown by first following the proof of (a) and finding a finite $\Gamma'_0 \subseteq_{fin} \Gamma'$ s.t.

$$\vdash_{b\Lambda}^{\mathcal{HR}} \Gamma'_0 \rightarrow \neg \exists z_1, \dots, z_{b\Lambda(\heartsuit)(i)}. (\bigwedge_{j \leq b\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \wedge x \heartsuit \dots [y_i : \bigvee_{j \leq b\Lambda(\heartsuit)(i)} y_i = z_j] \dots).$$

Applying BDPL_{bΛ} proves (b).

The condition (b) tells us that Γ itself does respect $Inf_{b\Lambda}$ by assumption. But if $|X| < \kappa$, then there are κ -many $z \in \mathbf{iVar}$ which are fresh for $\Gamma \cup X \cup \{\phi_1, \dots, \phi_{ar(\heartsuit)}\}$ and distinct from x and \bar{y} . For any tuple of such z 's, (b) would hold also for $\Gamma' = \Gamma \cup X$. This gives condition 3.2 for $Inf_{b\Lambda}$. \square

Claim 3.23. *Assume $|\mathbf{iVar}| = \kappa \geq |\Lambda \cup \Sigma \cup \omega|$. Then any $\vdash_{b\Lambda}^{\mathcal{HR}}$ -consistent set of formulas Γ s.t. $|\{x \in \mathbf{iVar} \mid x \text{ fresh for } \Gamma\}| = \kappa$ (in particular, a consistent set of sentences) can be extended to a maximally $\vdash_{b\Lambda}^{\mathcal{HR}}$ -consistent set of formulas Γ' s.t.*

- whenever $\exists x. \phi \in \Gamma'$, then $\phi[z/x] \in \Gamma'$ for some $z \in \mathbf{iVar}$
- whenever $\exists x. (x \heartsuit [x : \phi_1] \dots [x : \phi_n] \wedge \neg x \heartsuit [x : \psi_1] \dots [x : \psi_n]) \in \Gamma'$, then there is $z \in \mathbf{iVar}$ and $i \leq n$ s.t. $\neg(\phi_i \leftrightarrow \psi_i)[z/x] \in \Gamma'$.
- whenever $x \heartsuit \dots [y_i : \phi_i] \dots \in \Gamma'$ and $b\Lambda(\heartsuit)(i) \neq \infty$, then there are $z_1, \dots, z_{b\Lambda(\heartsuit)(i)}$ s.t. $x \heartsuit \dots [y_i : \bigvee_{j \leq b\Lambda(\heartsuit)(i)} y_i = z_j] \dots \in \Gamma'$ and moreover $\phi_i[z_j/y_i] \in \Gamma'$ for each $j \leq b\Lambda(\heartsuit)(i)$.

³In fact, a variant of the Generalization Theorem is available even for non-empty contexts as long as the quantified variable does not occur freely therein, cf. [End01, p. 117].

Proof. This immediately follows from the preceding Claim, Theorem 3.17 and the fact Γ' is a MCS. \square

Proof of Theorem 3.15. Recall Definition 3.19. We will build our quasi-Henkin model using Γ' . Satisfaction of condition 3.3 follows then directly from the first item in Claim 3.23, i.e., from being closed under $\text{Inf}_{\text{NAME}\alpha}$. Hence, we just need to define a transition structure γ on $C_{\Gamma'}$ and for this purpose, we need to find for each $|x|$ a suitable $t \in TC_{\Gamma'}$ s.t. when $\gamma(|x|)$ is defined as t , the condition 3.4 is satisfied. This is of course where we use the notions of one-step satisfiability and $\text{b}\Lambda\text{-SISC}$: we find this t in the non-empty intersection of the denotation of a certain one-step consistent subset of $\text{Rank1}(\text{sVar})$ under a suitable valuation of schematic variables in $C_{\Gamma'}$.

Assume then Rank1 has enough schematic variables to name all elements of $\text{CPL}(\Lambda, \Sigma)$; let \mathbf{p}_ϕ be the schematic variable corresponding to ϕ under some fixed assignment. For each x , we can define an evaluation $\tau_x(\mathbf{p}_\psi) = \hat{\psi}^x$. Note that for each pair of distinct x and y we have that $\tau_x(\mathbf{p}_{x=y})$ is a singleton, thanks to the definition of $C_{\Gamma'}$.

Thus, let us define for each $x \in \text{iVar}$ the set

$$\Psi_x := \{\epsilon \heartsuit \mathbf{p}_{\psi_1^\circ} \dots \mathbf{p}_{\psi_n^\circ} \mid \epsilon(x \heartsuit [x:\psi_1^\circ] \dots [x:\psi_n^\circ]) \in \Gamma'\},$$

where ϵ is either nothing or negation and for each $i \leq \text{ar}(\heartsuit)$, ψ_i° is either:

- ψ_i itself, if $\text{b}\Lambda(\heartsuit)(i) = \infty$ or
- $\bigvee_{j \leq \text{b}\Lambda(\heartsuit)(i)} x = z_j$ otherwise, where $z_1, \dots, z_{\text{b}\Lambda(\heartsuit)(i)}$ are s.t.
 - $x \heartsuit \dots [x : \bigvee_{j \leq \text{b}\Lambda(\heartsuit)(i)} x = z_j] \dots \in \Gamma'$ and moreover
 - $\psi_i[z_j/x] \in \Gamma'$ for each $j \leq \text{b}\Lambda(\heartsuit)(i)$.

Furthermore, let us define a colouring \mathbf{c}_x of schematic variables which assigns *fin* to every $\mathbf{p}_{\bigvee_{j \leq m} x = z_m}$, where z_1, \dots, z_m is any finite sequence of variables and ∞ to every other \mathbf{p}_ψ . It is clear that τ_x respects \mathbf{c}_x . The task of showing that Ψ_x is \mathbf{c}_x -consistent wrt τ_x , i.e., that

$$\Psi_x \cup \{\mathbf{P}\sigma \mid \sigma : \text{sVar} \rightarrow \text{Prop and } \mathbf{A}/\mathbf{P} \in \text{b}\Lambda_{\mathbf{c}}(\mathcal{R}) \text{ s.t. } C, \tau_x \models \mathbf{A}\sigma\} \not\models_{\text{CPC}} \perp.$$

using Lemma 3.4 is left to the reader; the basic ideas are available in several references such as [Sch07], [SP10c] and [SP10b]. \square

Remark 3.24. The similarities and differences between coalgebraic predicate logic and languages like $\text{H}_\Lambda(@)$ and its extensions to be discussed in §4 are best appreciated by comparing the proof of Theorem 3.15 with earlier announced results for coalgebraic hybrid logic [SP10b]. In the predicate case:

- not only one-step rules, but also non-standard naming and pasting rules of [SP10b] can be expressed as ordinary first-order axioms.
- As we are going to discuss now, Henkin-style completeness proof directly leads to the Omitting Types theorem. Nothing like this seems to hold for a language like $\text{H}_\Lambda(@)$ studied in [SP10b]; the presence of binding and/or quantification mechanism seems essential in the proof. Recall again that the presence of such mechanism also allowed us to reuse (equivalence classes of) variables as building block of models instead of Henkin-style constants.

3.4. Omitting Types Theorem. The Omitting Types Theorem is a standard result of model theory. Goldblatt [Gol93, §8.2] shows how to establish it using the Abstract Henkin Principle. Here is a more detailed description how to obtain it in our setting. In this section, we assume that the entire $\text{CPL}(\Lambda, \Sigma)$ is countable and we keep these countable Λ and Σ fixed and implicit.

Fix a finite subset of iVar $\{x_1, \dots, x_k\}$ and denote the set of all formulas whose free variables are contained in $\{x_1, \dots, x_k\}$ as $\text{CPL}(k)$. Thus, the set of sentences can be written as $\text{CPL}(0)$. Recall that a k -type (sometimes called a *complete type*) is a maximal consistent subset of $\text{CPL}(k)$. For any given $\Gamma \subseteq \text{CPL}(0)$ and any k -type Σ , say that Σ is *principal over* Γ if there is $\phi \in \Sigma$ s.t. $\forall \psi \in \Sigma. \Gamma \vdash_{\text{b}\Lambda}^{\mathcal{HR}} \phi \rightarrow \psi$. Say that a model $\mathfrak{M} = (C, \gamma, I)$ *realizes* k -type Σ if $\bigcap_{\psi \in \Sigma} \llbracket \psi \rrbracket_C^{x_1, \dots, x_k} \neq \emptyset$, where as before

$$\llbracket \phi \rrbracket_C^{x_1, \dots, x_k} = \{(c_1, \dots, c_k) \in C \mid \mathfrak{M}, v[c_1/x_1] \dots [c_k/x_k] \models \phi\};$$

a k -type is *omitted* by \mathfrak{M} if it is not realized by it.

Note that one consequence of being non-principal is that Σ is neither entailed by Γ nor inconsistent with it.

Theorem 3.25 (Omitting Types). *Whenever a set of rules \mathcal{R} is $\text{b}\Lambda$ -S1SC for a Λ -structure over T that is adequate for $\text{b}\Lambda$, Γ is a consistent set of sentences and Σ is a k -type non-principal over Γ , Γ has a model omitting Σ .*

Proof. We only need to refine somewhat the proof of the completeness theorem by using a richer set of inferences than *Inf*. Consider

$$\text{Inf}_\Sigma = \text{Inf} \cup \{ \langle \{ \sigma[z_1 \dots z_k / x_1 \dots x_k] \mid \sigma \in \Sigma \}, \perp \rangle \mid z_1, \dots, z_k \text{ distinct els. of } \text{iVar} \}$$

(we have not formally defined simultaneous substitution, but it should be clear how to extend conventions introduced in §2). We claim that the condition 3.2 of Theorem 3.17 is satisfied with $\kappa = \omega$. For assume it is not. Then there exists a finite set $\Delta \subseteq \text{CPL}$ and a finite tuple of variables z_1, \dots, z_k s.t. (*) $\Gamma \cup \Delta$ is consistent but

$$\Gamma \vdash_{\text{b}\Lambda}^{\mathcal{HR}} \bigwedge \Delta \rightarrow \sigma[z_1 \dots z_k / x_1 \dots x_k] \text{ for every } \sigma \in \Sigma.$$

Let $\overline{z'}$ be a sequence containing all the variables in Δ different from $z_1 \dots z_k$ and $\delta = \exists \overline{z'}. \bigwedge \Delta$. Then we have

$$(**) \Gamma \vdash_{\text{b}\Lambda}^{\mathcal{HR}} \delta \rightarrow \sigma[z_1 \dots z_k / x_1 \dots x_k] \text{ for every } \sigma \in \Sigma$$

and consequently, setting δ' to be $\delta[x_1 \dots x_k / z_1 \dots z_k]$

$$(***) \Gamma \vdash_{\text{b}\Lambda}^{\mathcal{HR}} \delta' \rightarrow \sigma \text{ for every } \sigma \in \Sigma$$

(in deriving (***) and (***) we obviously use the fact that Γ is a set of *sentences*).

As Σ is not principal over Γ and $\delta' \in \text{CPL}(k)$, we have that $\delta' \notin \Sigma$, hence $\neg \delta' \in \Sigma$. By (***) this means that $\Gamma \vdash_{\text{b}\Lambda}^{\mathcal{HR}} \neg \delta'$, and using again renaming and the fact that Γ is a set of sentences, we obtain a contradiction with (*). The rest proceeds as in the completeness proof. \square

Remark 3.26. Goldblatt [Gol93, §8.2] points out this can be extended to simultaneously omitting a countable set of non-principal types.

An application of the above theorem can be used to show, for example that CPL-theories over countable $\mathfrak{b}\Lambda$ -S1SC structures allowing infinite branching also have globally finitely branching models—and similar examples can be constructed, e.g., concerning boundedness of neighbourhood-like operators. It is worth contrasting with the proof of the incompleteness result (Theorem 3.33) we are going to present next.

3.5. ω -boundedness and Failure of Completeness. In this subsection, we show that there is a substantial gap between S1SC and finitary S1SC as conditions allowing for strong completeness, by proving that within a larger class of ω -bounded structures, the bounded structures are the only ones that satisfy compactness. Here, ω -boundedness of an operator means informally that its satisfaction can always be established by looking only at a finite subset of the successors, without however requiring a fixed bound on their number. In examples for this property, we concentrate on cases additionally satisfying *finitary one-step compactness*, a condition that would similarly be seen as essentially necessary for overall compactness and that will moreover become important in our forays into model theory (§ 5). In the whole subsection, to keep things simple we work with unary $\heartsuit \in \Lambda$.

Definition 3.27. A Λ -structure is *finitary one-step compact* if for every set X , every finitely satisfiable set $\Phi \subseteq \text{Prop}(\Lambda(\mathcal{P}_{\text{fin}}(X)))$ of one-step formulas is satisfiable.

Remark 3.28. Finitary one-step compactness is clearly a consequence of finitary S1SC, hence all our “Kripke-like” cases enjoy this property.

Definition 3.29 (ω -Bounded operators). A modal operator \heartsuit is ω -bounded if for each set X and each $A \subseteq X$,

$$\llbracket \heartsuit \rrbracket_X(A) = \bigcup_{B \subseteq_{\text{fin}} A} \llbracket \heartsuit \rrbracket_X(B).$$

Example 3.30 (Nonstandard subdistributions). We generally write \mathcal{S} for the discrete subdistribution functor, i.e. $\mathcal{S}(X)$ consists of real-valued discrete measures μ on X such that $\mu(X) \leq 1$, and for maps f , $\mu(f)$ takes image measures. As a variant of this functor, we consider the discrete subdistributions functor \mathcal{S}^{rc} where measures take values in real-closed fields. Explicitly: we intend to model Markov chains with non-standard probabilities; these consist of a set X of states, and at each state x an R_x -valued transition distribution μ_x , where R_x is a real-closed field (i.e. a model of the first-order theory of the reals). These structures are coalgebras for the functor T which maps a set X to the set of pairs (R, μ) where R is a real-closed field and μ is an R -valued discrete subdistribution on X (again meaning that $\mu(X) \leq 1$). This functor is in fact class-valued, which however does not affect the applicability of our coalgebraic analysis (which never requires iterated application of the coalgebraic type functor, e.g. it does not use the terminal sequence). We take the modal signature Λ to consist of the operators $\langle p \rangle$ (‘with probability more than p ’) for $p \in [0, 1] \cap \mathbb{Q}$.

We show that the $\langle p \rangle$ are ω -bounded and that the arising logic \mathcal{L} is finitary one-step compact. To see the former, let $(R, \mu) \in TX$ and let $A \subseteq X$ such that $\mu \models \langle p \rangle A$, i.e. $\sum_{x \in A} \mu(x) > p$. Then there exists $B \subseteq_{\text{fin}} A$ such that $\sum_{x \in B} \mu(x) > p$, i.e. $\mu \models \langle p \rangle B$. Since $\langle p \rangle$ is clearly monotone, this implies that $\llbracket \langle p \rangle \rrbracket_X(A) = \bigcup_{B \subseteq_{\text{fin}} A} \llbracket \langle p \rangle \rrbracket_X(B)$, as required.

To show that \mathcal{L} is finitary one-step compact, let $\Phi \subseteq \text{Prop}(\Lambda(\mathcal{P}_{\text{fin}}(X)))$ be finitely satisfiable. Extend the standard language of real arithmetic with a constant symbol c_x for each

element of X , obtaining a language L . Then satisfaction of a formula in $\text{Prop}(\Lambda(\mathcal{P}_{fin}(X)))$ by $\mu \in TX$ translates into a first-order formula over L with c_x representing $\mu(x)$; specifically, the translation t commutes with the Boolean connectives and translates formulas $\langle p \rangle B$ with $A \in \mathcal{P}_{fin}(X)$ into $\sum_{x \in A} c_x > p$. Applying t to Φ and introducing additional formulas $c_x \geq 0$ for all $x \in X$ and $\sum_{x \in A} c_x \leq 1$ for all $A \in \mathcal{P}_{fin}(X)$ thus produces a finitely satisfiable, and hence satisfiable, set of first-order formulas over L . A model of this set consists of a real-closed field R and interpretations $\hat{c}_x \in R$ of the constants c_x such that putting $\mu(x) = \hat{c}_x$ defines a discrete subdistribution (note that $\sum_{x \in A} \hat{c}_x \leq 1$ for all $A \in \mathcal{P}_{fin}(X)$ implies $\sum_{x \in X} \hat{c}_x \leq 1$), which then yields a model (R, μ) of Φ .

Example 3.31 (Zero-dimensional subdistributions). Fix a zero-dimensional closed (hence compact) subset $Z \subseteq [0, 1]$, e.g. a discrete set or the Cantor space, and let \mathcal{S}^Z be the associated *zero-dimensional discrete subdistributions functor*, i.e. the subfunctor of the subdistribution functor \mathcal{S} where probabilities of finite sets of states are restricted to take values in Z :

$$\mathcal{S}^Z(X) = \{\mu \in \mathcal{S}(X) \mid \forall A \in \mathcal{P}_{fin}(X). \mu(A) \in Z\}.$$

Moreover, we restrict the probabilities p in operators $\langle p \rangle$ to be such that $(p, 1] \cap Z$ is clopen in Z ; since Z is zero-dimensional, there exist enough such p to separate all values in Z . As before, all these operators are ω -bounded. It remains to show that the logic is finitary one-step compact. So let $\Phi \subseteq \text{Prop}(\Lambda(\mathcal{P}_{fin}(X)))$ be finitely satisfiable. Note that the space Z^X , equipped with the product topology, is compact. We equip $\mathcal{S}^Z(X)$ with the subspace topology in Z^X . Observe that the condition $\forall A \in \mathcal{P}_{fin}(X). \mu(A) \in Z$ already implies $\mu(X) \leq 1$; since for $A \in \mathcal{P}_{fin}(X)$, the summation map $Z^A \rightarrow Z$ is continuous (this would fail for infinite A), it follows that $\mathcal{S}^Z(X)$ is closed in Z^X , hence compact.

By the restriction placed on the indices p in modal operators $\langle p \rangle$, and again using continuity of finite summation, we have that for every formula $\langle p \rangle A$ with $A \in \mathcal{P}_{fin}(X)$, the extension

$$\llbracket \langle p \rangle A \rrbracket = \{\mu \in \mathcal{S}^Z(X) \mid \mu(A) > p\}$$

is clopen in $\mathcal{S}^Z(X)$. As clopen sets are closed under Boolean combinations, we thus have that the extension of every formula in $\text{Prop}(\Lambda(\mathcal{P}_{fin}(X)))$ is clopen in $\mathcal{S}^Z(X)$. Let \mathfrak{A} denote the family of clopens induced in this way by formulas in Φ . Finite satisfiability of Φ implies that \mathfrak{A} has the finite intersection property, and hence has non-empty intersection by compactness of $\mathcal{S}^Z(X)$. It follows that Φ is satisfiable.

To formalize our incompleteness result, we require the following notion of *propositional atom*:

Definition 3.32. A nullary modality $p \in \Lambda$ is a *propositional atom* if T decomposes as $T = T' \times 2$ and under this decomposition, $\llbracket p \rrbracket_X = T'X \times \{\top\}$.

Almost all modalities we have encountered in our examples will standardly be combined with propositional atoms. Formally, if V is a set of propositional atoms and T' is a functor, then the atoms $p \in V$ give rise to nullary modalities p , interpreted over $T = T' \times \mathcal{P}(V)$ by $\llbracket p \rrbracket_X = \{(t, U) \in T'X \times \mathcal{P}(V) \mid p \in U\}$.

Theorem 3.33. *Whenever a Λ -structure makes some $\heartsuit \in \Lambda$ ω -bounded without being k -bounded for any $k \in \omega$, strong completeness fails whenever either Σ contains a predicate symbol of positive arity or Λ contains a propositional atom.*

Proof. Assume $P \in \Sigma$ is a predicate symbol of positive arity, w.l.o.g. unary, and $\heartsuit \in \Lambda$ is as in the statement of theorem. Consider

$$\Gamma = \{x\heartsuit[y : P(y)]\} \cup \{\forall y_1, \dots, y_k. (P(y_1) \wedge \dots \wedge P(y_k) \rightarrow \neg x\heartsuit[y : y = y_1 \vee \dots \vee y = y_k]) \mid k \in \omega\}.$$

Clearly, every finite subset of Γ is satisfiable in a model based on a coalgebra witnessing the failure of k -boundedness for a suitably large k . However, a coalgebraic model satisfying the whole Γ would witness the failure of ω -boundedness. This means that Γ is a counterexample to compactness, and hence no finitary deduction system can be strongly complete. The proof for the case where Λ contains a propositional atom is entirely analogous. \square

Example 3.34. The probabilistic instances of CPL given by interpreting the probabilistic modalities $\langle p \rangle$ over nonstandard or zerodimensional subdistributions, respectively, and are ω -bounded but fail to be k -bounded for any k . Hence they fail to be compact by Theorem 3.33 (once equipped with propositional atoms) although they satisfy finitary one-step compactness (Examples 3.30 and 3.31).

4. CORRESPONDENCE WITH COALGEBRAIC MODAL LOGIC

We next compare the expressivity of CPL with that of various coalgebraic modal and hybrid logics.

4.1. Coalgebraic Standard Translation for CML. The formulas $\text{CML}_\Lambda \Sigma$ of pure (coalgebraic) modal logic in the modal signature Λ over Σ (now all elements of Σ are assumed to be of arity 1) are given by the grammar:

$$\text{CML}_\Lambda \Sigma \quad \phi, \psi ::= P \mid \perp \mid \phi \rightarrow \psi \mid \heartsuit(\phi_1, \dots, \phi_n),$$

where $P \in \Sigma$.

Satisfaction is defined with respect to $\mathfrak{M} = (C, \gamma, I)$ and a specific point $c \in C$ in a standard way, see e.g. [SP10a, SP10b].

Definition and Proposition 4.1. *Define the coalgebraic standard translation as*

$$\begin{aligned} ST_x(P) &= P(x), \\ ST_x(\heartsuit(\phi_1, \dots, \phi_n)) &= x\heartsuit[x : ST_x(\phi_1)] \dots [x : ST_x(\phi_n)], \\ ST_x(\perp) &= \perp, \\ ST_x(\phi \rightarrow \psi) &= ST_x(\phi) \rightarrow ST_x(\psi). \end{aligned}$$

Then for any $\phi \in \text{CML}_\Lambda \Sigma$ and any $\mathfrak{M} = (C, \gamma, I), v, c$, we have $\mathfrak{M}, c \models \phi$ iff $\mathfrak{M}, v[x \mapsto c] \models ST_x(\phi)$.

For example, $ST_x(\heartsuit\heartsuit P) = x\heartsuit[x : x\heartsuit[x : P(x)]]$. This definition is more straightforward than the standard translation into FOL of modal logic over ordinary Kripke frames. Moreover, ST_x uses only one variable from iVar , namely x itself. In fact, we can immediately observe that

Proposition 4.2. *Whenever Σ consists entirely of unary predicate symbols, the subset of $\phi \in \text{CPL}(\Sigma)$ obtained as the image of ST_x for a fixed $x \in \text{iVar}$ consists precisely of equality-free and quantifier-free formulas in the variable x .*

4.2. Hybrid Languages. In this section, we establish the equivalence of CPL with the hybrid languages $H_\Lambda(\downarrow, \mathbf{A})$ and $H_\Lambda(\forall, @)$. Both correspondences also hold for ordinary predicate logic over relational structures (FOL) and extend to CPL. We take this as yet another indication that CPL is natural and well-designed both as a generalization of FOL and “the” predicate logic cousin of existing coalgebraic formalisms.

This is our main, but not the only motivation. We progress towards this result step-by-step, extending the modal language gradually with new hybrid constructs. In this way, we reveal that a similar correspondence exists between natural fragments of CPL and weaker hybrid languages, most importantly between quantifier-free CPL and $H_\Lambda(\downarrow, @)$.

Again, obviously the correspondence between fragments of CPL and extensions of CML is tighter than in the case of FOL and ML only due to the modal flavour of CPL. However, results such as Corollary 4.5 are useful spadework: any model-theoretic tool to be developed—say, a variant of E-F games—would be adequate for an extended coalgebraic modal formalism (e.g., $H_\Lambda(\downarrow, @)$) **iff** it is adequate for the corresponding fragment of CPL (e.g., the variable-free fragment), so we are free to work with whichever formalism we find more convenient at a given moment. The straightforward correspondence also provides a good starting point for an extension of research programme sketched in [Cat05]—see Remark 4.9 at the end of this section.

Given a supply of *world variables* \mathbf{wVar} that we are going to keep fixed and implicit—in fact, as stated below, *near* identical to \mathbf{iVar} —we define the following *coalgebraic hybrid languages*

$$\begin{array}{lll} H_\Lambda(\downarrow, @) & \phi, \psi ::= & z \mid P \mid \perp \mid \phi \rightarrow \psi \mid \heartsuit(\phi_1, \dots, \phi_n) \mid @_z \phi \mid \downarrow z. \phi \\ H_\Lambda(\downarrow, \mathbf{A}) & \phi, \psi ::= & z \mid P \mid \perp \mid \phi \rightarrow \psi \mid \heartsuit(\phi_1, \dots, \phi_n) \mid \mathbf{A} \phi \mid \downarrow z. \phi \\ H_\Lambda(\forall, @) & \phi, \psi ::= & z \mid P \mid \perp \mid \phi \rightarrow \psi \mid \heartsuit(\phi_1, \dots, \phi_n) \mid @_z \phi \mid \forall z. \phi \end{array}$$

where $z \in \mathbf{wVar}$. We refer the reader to, e.g., [SP10b, BC06, Cat05] for the semantics. The extension of the standard translation to these formalism is unproblematic in some cases, just like in the case of ordinary hybrid logic over Kripke frames:

$$ST_x(z) = x = z, \quad ST_x(\mathbf{A} \phi) = \forall x. ST_x(\phi), \quad ST_x(\forall z. \phi) = \forall z. ST_x(\phi).$$

One is tempted to put forward also

$$ST_x(@_z \phi) = ST_x(\phi)[z/x], \quad ST_x(\downarrow z. \phi) = ST_x(\phi)[x/z].$$

However, with other clauses remaining the same, this would violate our convention that $[z/x]$ is used only when z is *substitutable* for x ; we would need to interpret it as *capture-avoiding* substitution. Sadly, this in turn would entail forsaking the luxury of using just one designated variable for comprehension. Guillaume Malod (see [CF05]) observed that if we restrict the supply of variables, a translation along the above lines—indeed first proposed in the literature, which also goes to show that the present discussion is less trivial than it might seem—would fail even when embedding the hybrid logic over Kripke frames in the two-variable fragment of FOL. Malod’s counterexample used nesting of modalities of level two, but as our translation uses just one designated variable, ST would go wrong already on formulas of depth one. Just consider $ST_x(\downarrow z. \Diamond z)$: were we careless about capture of bound variables, we would obtain $x \Diamond [x : x = x]$, which is a formula with a completely different meaning. There are two ways out. First is to redefine

Table 2: Coalgebraic Hybrid Translation from quantifier-free CPL to $H_\Lambda(\downarrow, @)$

$$\begin{array}{ll}
HT(P(x)) = @_x P & HT(x = y) = @_x y \\
HT(\perp) = \perp & HT(\phi \rightarrow \psi) = HT(\phi) \rightarrow HT(\psi) \\
xHT(x \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n]) & = @_x \heartsuit (\downarrow y_1. HT(\phi_1), \dots, \downarrow y_n. HT(\phi_n))
\end{array}$$

$$STmod_x(@_z \phi) = \forall x. (x = z \rightarrow ST_x(\phi)), \quad (4.1)$$

$$STmod_x(\downarrow z. \phi) = \forall z. (x = z \rightarrow ST_x(\phi)). \quad (4.2)$$

The second is to keep ST for hybrid formulas as defined above and change the modal clause instead:

$$ST_x(\heartsuit(\phi_1, \dots, \phi_n)) = x \heartsuit [y : ST_y(\phi_1)] \dots [y : ST_y(\phi_n)], \quad (4.3)$$

where y is the first (in some fixed enumeration) variable *not used* in $ST_x(\phi_1), \dots, ST_x(\phi_n)$; by *not used* here we mean both free and bound usage. Furthermore, to ensure that the translation works correctly, we have to assume that *neither x nor y* appears in \mathbf{wVar} . While the requirement to use more bound variables can be cumbersome—particularly for infinite sets of formulas—we prefer this option, as it makes it easier to characterize weaker hybrid languages as suitable syntactic fragments of CPL.

We can now state a generalization of both Proposition 4.1 and corresponding results from the hybrid logic literature—see, e.g., [BC06] for references:

Proposition 4.3. *For any hybrid formula ϕ and any $\mathfrak{M} = (C, \gamma, I), v, c$, we have $\mathfrak{M}, v, c \models \phi$ iff $\mathfrak{M}, v[x \mapsto c] \models ST_x(\phi)$.*

As is well-known in the hybrid logic community—see again [BC06] for references—there is also a translation in the reverse direction for sufficiently expressive hybrid languages. This also generalizes to our setting, see Table 2.

Proposition 4.4. *For any $\phi \in \text{CPL}$ and any $\mathfrak{M} = (C, \gamma, I), v, c$, we have*

$$\mathfrak{M}, v, c \models HT(\phi) \text{ iff } \mathfrak{M}, v[x \mapsto c] \models \phi.$$

Combining Propositions 4.4 and 4.3, we get:

Corollary 4.5. *Whenever Σ consists purely of unary predicates (and no function symbols), $H_\Lambda(\downarrow, @)$ is expressively equivalent to the quantifier-free fragment of CPL, assuming \mathbf{iVar} contains \mathbf{wVar} plus a disjoint infinite supply of additional individual variables (used for comprehension).*

Remark 4.6 (Quantifier-free CPL as the bounded fragment of FOL). In the case of ordinary FOL, the fragment equivalent to $H_\Lambda(\downarrow, @)$ is characterized as the *bounded fragment*, see, e.g., [AtC07]. In fact, our formula $x \heartsuit [y : \phi]$, despite being quantifier-free on the surface, can be described as a form of bounded quantification. This can be formalized as a result stating that over coalgebras for the covariant powerset functor (Kripke frames), quantifier-free CPL is equivalent to the bounded-fragment of ordinary FOL, where the role of \heartsuit in CPL is played by the binary relation symbol R in FOL; details are left to the reader.

Remark 4.7 (Chang’s original syntax). As already mentioned, our syntax is slightly different to the original one proposed by Chang [Cha73]. In that paper, there were no explicit

comprehension variables and even in the enriched syntax which allowed constants and function terms, the term on the left-hand side of \heartsuit had to be a variable. This variable was reused then on the right side of \heartsuit as the comprehension variable. In other words, Chang's $x\heartsuit\phi(x)$ was equivalent to ours $x\heartsuit[x : \phi(x)]$. In presence of quantifiers, which can be used to simulate the effect of capture-avoiding substitution as in *STmod* (this trick in fact stems back to Alfred Tarski), the two languages are obviously equivalent. But when considering fragments, as we do here, the equivalence breaks down; without quantifiers, Chang's syntax does not allow (4.2) and simple renaming of the comprehension variable on the right-hand side of \heartsuit as in (4.3) is not possible either.

There are two usual routes in hybrid logic to achieve full first-order expressivity. One is to add universal quantifiers over \mathbf{wVar} in presence of the satisfaction operator $@$. The other is to add the global modality A in presence of the downarrow binder \downarrow . The hybrid translation is extended then as follows:

$$\begin{aligned} HT_{\forall @}(\forall x.\phi) &= \forall x.HT(\phi) \\ HT_{A\downarrow}(\forall x.\phi) &= \downarrow y.A \downarrow x.A(y \rightarrow \phi) \end{aligned}$$

In $HT_{A\downarrow}$ we need the proviso that y is not occurring in the whole formula.

Theorem 4.8. $H_\Lambda(\downarrow, A)$, $H_\Lambda(\forall, @)$ and CPL are expressively equivalent.

As we can use $STmod_x$ now and keep reusing x as the comprehension variable, it is enough to assume that $\mathbf{iVar} = \mathbf{wVar} \cup \{x\}$. Since $@_z\phi$ is definable in presence of A (as $A(z \rightarrow \phi)$), \downarrow is definable by the universal quantifier over \mathbf{wVar} (as $\forall z.(z \rightarrow \phi)$) and A is definable by combination of \forall and $@$ (as $\forall y.@_y\phi$, where y is not used in ϕ), we get in fact seven equivalent languages: CPL, Chang's original language, $H_\Lambda(\downarrow, A)$, $H_\Lambda(\forall, @)$, $H_\Lambda(\downarrow, A)$ with $@$, $H_\Lambda(\forall, @)$ with \downarrow and the jumbo hybrid language with all connectives introduced above.

Remark 4.9. The equivalences stated here extend to the case of hybrid languages and CPL enriched with quantification over predicates (i.e., second-order languages). It would be interesting to follow more thoroughly the program of *coalgebraic abstract model theory* both above and below CPL. See Ten Cate's PhD Thesis [Cat05] for spadework in abstract model theory below first-order logic.

4.3. Semantic Correspondence: The Van Benthem-Rosen Theorem. Our Proposition 4.2 provides a *syntactic* characterization of the modal fragment of our language. In a companion paper [SPLar], we develop a *semantic*, Van Benthem-Rosen style characterization. To compare these two characterizations, let us briefly recall the details.

In the context of standard Kripke models, expressiveness of modal logic is characterized by van Benthem's theorem: modal logic is the bisimulation invariant fragment of first-order logic in the corresponding signature. The finitary analogue of this theorem [Ros97] states that every formula that is bisimulation invariant *over finite models* is equivalent *over finite models* to a modal formula. In the coalgebraic context, replace bisimilarity with behavioural equivalence [Sta11]. Moreover, we need to assume that the language has 'enough' expressive power; e.g., we cannot expect that bisimulation invariant formulas are equivalent to CML formulas over the empty similarity type. This is made precise as follows:

Definition 4.10. A Λ -structure is *separating* if, for every set X , every element $t \in TX$ is uniquely determined by the set $\{(\heartsuit, A) \mid \heartsuit \in \Lambda \text{ } n\text{-ary}, A \in \mathcal{P}(X)^n, t \in \llbracket \heartsuit \rrbracket_X(A)\}$.

Separation is in general a less restrictive condition than those we needed for completeness proofs. In particular, separation automatically obtains for Kripke semantics. It was first used to establish the Hennessy-Milner property for coalgebraic logics [Pat04, Sch08].

Theorem 4.11 ([SPLar]). *Suppose that the structure is separating and $\phi(x)$ is a CPL formula with one free variable. Then ϕ is invariant under behavioural equivalence (over finite models) iff it is equivalent to an infinitary CML formula with finite modal rank (over finite models).*

If we deal with finite similarity types only, the conclusion can be strengthened:

Theorem 4.12 ([SPLar]). *Suppose that the structure is separating, Λ is finite and $\phi(x)$ is a CPL formula with one free variable. Then ϕ is invariant under behavioural equivalence (over finite models) iff ϕ is equivalent to a **finite** CML formula (over finite models).*

In fact, we can combine Theorem 4.12 with the syntactic characterization of Proposition 4.2 to obtain

Corollary 4.13. *Whenever Σ consists entirely of unary predicate symbols (and there are no function symbols) and the structure is separating, the behaviourally-invariant (over finite structures) formulas of CPL in one-free variable are up to equivalence (over finite structures) precisely the equality-free and quantifier-free formulas in the single-variable fragment of CPL.*

5. FIRST STEPS IN COALGEBRAIC MODEL THEORY

We proceed to outline the beginning of coalgebraic model theory, taking a look at ultraproducts and the downwards Löwenheim-Skolem theorem.

Recall that if \mathcal{U} is an ultrafilter on an index set I and (X_i) is an I -indexed family of sets, then the *ultraproduct* $\prod_{\mathcal{U}} X_i$ is defined as

$$\prod_{\mathcal{U}} X_i = (\prod_{i \in I} X_i) / \sim$$

where \sim is the equivalence relation on $\prod_{i \in I} X_i$ defined by

$$(x_i) \sim (y_i) \iff \{i \in I \mid x_i = y_i\} \in \mathcal{U}.$$

One may regard \mathcal{U} as a $\{0, 1\}$ -valued measure on I ; under this reading, the above definition says that (x_i) and (y_i) are identified under \sim if they are almost everywhere equal. We write elements of $\prod_{\mathcal{U}} X_i$ and $\prod_{i \in I} X_i$ just as x , omitting notation for equivalence classes and accessing the i -th component as x_i .

Observe that if $X = \prod_{\mathcal{U}} X_i$ is an ultraproduct of sets and (A_i) is a family of subsets $A_i \subseteq X_i$, then

$$A = \prod_{\mathcal{U}} A_i := \{x \mid \{i \mid x_i \in A_i\} \in \mathcal{U}\} \quad (5.1)$$

is a well-defined subset of X (this is in fact just the way unary predicates are standardly extended from the components to the ultraproduct). Subsets of ultraproducts that are of this form are called *admissible*.

Lemma 5.1. *All finite subsets of ultraproducts are admissible.*

Ultraproducts of coalgebras will not be determined uniquely; instead, we give a property-oriented definition of and later show existence.

Definition 5.2 (Quasi-Ultraproducts of Coalgebras). Let $(C_i) = (X_i, \xi_i)_{i \in I}$ be a family of T -coalgebras, and let \mathfrak{U} be an ultrafilter on I . A coalgebra ξ on the set-ultraproduct $X = \prod_{\mathfrak{U}} X_i$ is called a *quasi-ultraproduct* of the C_i if for every family (A_i) of subsets $A_i \subseteq X_i$, every $x \in \prod_{\mathfrak{U}} X_i$, and every $\heartsuit \in \Lambda$,

$$\xi(x) \in \llbracket \heartsuit \rrbracket_X \prod_{\mathfrak{U}} A_i \iff \{i \in I \mid \xi_i(x_i) \in \llbracket \heartsuit \rrbracket_{C_i}(A_i)\} \in \mathfrak{U}. \quad (5.2)$$

The notion of quasi-ultraproduct extends naturally to coalgebraic models using the standard definition to extend the interpretation of predicates (as indicated above, Equation (5.1) recalls the case of unary predicates).

The definition of quasi-ultraproducts is designed in such a way that Łoś's theorem, which in the measure-theoretic view of ultraproducts states that the ultraproduct satisfies exactly those formulas that hold in almost all its components, extends to coalgebras:

Theorem 5.3 (Coalgebraic Łoś's Theorem). *If $\mathfrak{M} = (C, \gamma, V)$ is a quasi-ultraproduct of $\mathfrak{M}_i = (C_i, \gamma_i, V_i)$ for the ultrafilter \mathfrak{U} , then for every tuple (a^1, \dots, a^n) of states in C , where $a^k = (a_i^k)_{i \in I}$, and for every CPL formula $\phi(x_1, \dots, x_n)$, $C \models \phi(a^1, \dots, a^n) \iff \{i \mid C_i \models \phi(a_i^1, \dots, a_i^n)\} \in \mathfrak{U}$.*

Proof. Induction over formulas. The cases for Boolean operators and quantifiers are as in the classical case, and the case for modal operators is exactly by the quasi-ultraproduct property. \square

From this theorem, we obtain the usual applications, in particular compactness. The question is, of course, when quasi-ultraproducts exist. A core observation is

Lemma 5.4. *In the notation of Definition 5.2, the demands placed on $\xi(x)$ by (5.2) constitute a finitely satisfiable set of one-step formulas.*

Proof. We consider finitely many instances of (5.2) for families of sets $(A_i^j)_{i \in I}$ and sets $A^j = \prod_{\mathfrak{U}} A_i^j$, $j = 1, \dots, k$. We regard these sets as extensions of unary predicates P^j over the X_i and over X , respectively. If the corresponding instances of (5.2) do not have a solution $\xi(x)$ in TX , then this unsolvability means that we have a sound one-step rule \mathbf{A}/\mathbf{P} over \mathbf{sVar} and a valuation $\tau : \mathbf{sVar} \rightarrow \{A^1, \dots, A^k\}$ such that $X \models \mathbf{A}\tau$ but the instances of (5.2) for A^1, \dots, A^k demand $\xi(x) \models \neg \mathbf{P}\tau$; w.l.o.g. $\mathbf{sVar} = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ and $\tau(\mathbf{p}_j) = A^j$ for $j = 1, \dots, k$. Then X satisfies the first-order sentence $\forall z. (\mathbf{A}\sigma)$ where $\sigma(a_j) = P^j(y)$. By Łoś's theorem (in fact already by its classical version), there exists $B \in \mathfrak{U}$ such that $X_i \models \forall z. (\mathbf{A}\sigma)$ and hence $X_i \models \mathbf{A}\tau_i$ for all $i \in B$, where $\tau_i(\mathbf{p}_j) = A_i^j$. By one-step soundness of \mathbf{A}/\mathbf{P} this implies $TX_i \models \mathbf{P}\tau_i$ for all $i \in B$. But our formulation above that the instances of (5.2) for A^1, \dots, A^k demand $\xi(x) \models \neg \mathbf{P}\tau$ means more explicitly (and using the fact that \mathfrak{U} is an ultrafilter) that $\{i \in I \mid \xi_i(x_i) \models \neg \mathbf{P}\tau_i\} \in \mathfrak{U}$, so that we have a contradiction. \square

From Lemma 5.4, our first existence criterion for quasi-ultraproducts is immediate:

Theorem 5.5. *If a Λ -structure is one-step compact, then it has quasi-ultraproducts.*

Example 5.6. The above criterion applies in particular to all neighbourhood-like logics. It thus subsumes Chang's original ultraproduct construction [Cha73]

Like for our completeness results, an alternative is to require bounded operators:

Theorem 5.7. *If a Λ -structure is finitary one-step compact and all its operators are bounded, then it has quasi-ultraproducts.*

The proof needs the following lemma.

Lemma 5.8. *Let $(C_i) = (X_i, \xi_i)_{i \in I}$ be a family of T -coalgebras, and let \mathfrak{U} be an ultrafilter on I . Let X be the ultraproduct $\prod_{\mathfrak{U}} X_i$, and let $x \in X$. Then the set*

$$\Psi = \{\epsilon \heartsuit \{y^1, \dots, y^k\} \mid \{i \mid \xi_i(x_i) \models \epsilon \heartsuit \{y_i^1, \dots, y_i^k\}\} \in \mathfrak{U}\}$$

of one-step formulas (where \heartsuit ranges over Λ , the y^i range over X , and ϵ stands for either negation or nothing) is finitely satisfiable.

Proof. Analogous to Lemma 5.4, using atoms of the form $z = c$ in place of unary predicates. In more detail: if a finite subset of Ψ is unsatisfiable, then this amounts to $TX \models \mathbf{W}$ for some conjunctive clause $\mathbf{W} \in \text{Prop}(\Lambda(\mathcal{P}_{\text{fin}}(X)))$; hence $X \models \mathbf{A}$ for some sound rule \mathbf{A}/\mathbf{P} (using finite subsets of X as variables directly). Now $X \models \mathbf{A}$ is semantically equivalent to $X \models \forall z. \mathbf{A}_0$ where \mathbf{A}_0 is propositional formula over atoms $z = c$, where c ranges over constants denoting elements of the involved finite subsets of X . Then $\{i \mid X_i \models \forall z. \mathbf{A}_0\} \in \mathfrak{U}$ by (the classical version of) Łoś's theorem, where we interpret constants in X_i by taking the i -th component of the interpretation in X (this is just the way the interpretation of constants in the factors relates to that in the ultraproduct, classically). Hence $\{i \mid X_i \models \mathbf{A}\sigma_i\} \in \mathfrak{U}$, where σ_i replaces $\{y^1, \dots, y^k\}$ with $\{y_i^1, \dots, y_i^k\}$, and hence $\{i \mid TX_i \models \mathbf{W}\} \in \mathfrak{U}$, contradiction as in Lemma 5.4. \square

Proof (Theorem 5.7). By Lemma 5.8 and finitary one-step compactness, there exists $\xi(x)$ satisfying the set Ψ from Lemma 5.8. To show 5.2 for $A \subseteq X$, we regard A as the extension of a unary predicate P . Then $\xi(x) \models \heartsuit A$ is equivalent to

$$x \models \exists y^1, \dots, y^k. (P(y^1) \wedge \dots \wedge P(y^k) \wedge x \heartsuit [z : z = y^1, \dots, z = y^k]).$$

Thus it suffices to prove the Łoś equivalence for open formulas $x \heartsuit [z : z = y^1, \dots, z = y^k]$. This, however, is exactly what satisfaction of Ψ by $\xi(x)$ guarantees. \square

For operators that are ω -bounded but not k -bounded for any k , the ultraproduct construction cannot be available, in consequence of Theorem 3.33. However, the downward Löwenheim-Skolem theorem does survive under the weaker assumption of ω -boundedness:

Theorem 5.9 (Downward Löwenheim-Skolem Theorem). *If a Λ -structure is ω -bounded and finitary one-step compact, then $\text{CPL}(\Lambda, \Sigma)$ satisfies the downward Löwenheim-Skolem theorem.*

The proof needs the following simple lemma.

Lemma 5.10. *Let Y be an infinite subset of X , $\tau : \text{sVar} \rightarrow \mathcal{P}_{\text{fin}}(Y)$ and $\mathbf{A} \in \text{Prop}(\text{sVar})$. Then $Y \models \mathbf{A}\tau$ iff $X \models \mathbf{A}\tau$.*

Proof. Only finitely many $\mathbf{p} \in \text{sVar}$ are relevant, so we can assume that sVar is finite. Define the τ -valuation of $x \in X$ as the valuation $\kappa : \text{sVar} \rightarrow 2$ given by $\kappa(\mathbf{p}) = \top$ iff $x \in \tau(\mathbf{p})$. Then the claim of the lemma is equivalent to saying that every τ -valuation occurring in X occurs also in Y . Now if $x \in X \setminus Y$, then the τ -valuation of x is everywhere false; this valuation occurs also in Y , as sVar and the $\tau(\mathbf{p})$ are finite. \square

Proof (Theorem 5.9). Let Φ be a set of coalgebraic first-order formulas in $\text{CPL}(\Lambda, \Sigma)$, and let $\mathfrak{M} = (X, \xi, V)$ be such that $\mathfrak{M} \models \Phi$. Pick Skolem functions for all occurrences of subformulas $\exists x. \phi$ in Φ as usual, and for every occurrence of a subformula $x \heartsuit [y : \phi]$ in Φ a finitely non-deterministic Skolem function $f_{x \heartsuit [y : \phi]} : X^{FV(x \heartsuit [y : \phi])} \rightarrow \mathcal{P}_{fin}(X)$ with the property that for every valuation $\eta \in X^{FV(x \heartsuit [y : \phi])}$, $f_{x \heartsuit [y : \phi]}(\eta) \subseteq_{fin} \llbracket \phi \rrbracket_{C, \eta}$ and

$$C, \eta \models x \heartsuit [y : \phi] \iff \xi(\eta(x)) \models \heartsuit f_{x \heartsuit [y : \phi]}(\eta).$$

(Such a function $f_{x \heartsuit [y : \phi]}$ exists because \heartsuit is ω -bounded.) Pick a countably infinite subset $Y_0 \subseteq X$ and let Y be the closure of Y_0 under the Skolem functions, in the case of the non-deterministic Skolem functions $f_{x \heartsuit [y : \phi]}$ in the sense that $f_{x \heartsuit [y : \phi]}[Y] \subseteq Y$. Then Y is countable: it consists of the possible values of countably many finitely non-deterministic finite Skolem terms.

It remains to define a coalgebra structure ζ on $y \in Y$ in such a way that

$$\zeta(y) \models \heartsuit A \iff \xi(y) \models \heartsuit A \quad (5.3)$$

for all $A \subseteq_{fin} Y$; that is, we have to prove that the set

$$\Psi = \{\epsilon \heartsuit A \mid \xi(y) \models \epsilon \heartsuit A\}$$

of one-step formulas over $\mathcal{P}_{fin}(Y)$ is satisfiable over Y (where \heartsuit ranges over Λ , A ranges over $\mathcal{P}_{fin}(Y)$, and ϵ ranges over $\{\cdot, \neg\}$). By finitary one-step compactness, it suffices to prove that Ψ is finitely satisfiable. Assume the contrary; then there exists a sound one-step rule \mathbf{A}/\mathbf{P} over V and a valuation τ for sVar taking values in $\mathcal{P}_{fin}(Y)$ such that $Y \models \mathbf{A}\tau$ and $\mathbf{P}\tau$ propositionally contradicts some finite subset Ψ_0 of Ψ . By Lemma 5.10, $X \models \mathbf{A}\tau$, and hence $X \models \mathbf{P}\tau$; therefore, Ψ_0 is unsatisfiable over X , in contradiction to the fact that $\xi(y)$ satisfies Ψ by construction.

Since Ψ is satisfiable, we have a coalgebra structure ζ satisfying (5.3). It follows by induction over the formula structure that for every coalgebraic first-order formula ϕ and every valuation η in Y ,

$$(Y, \zeta), \eta \models \phi \quad \text{iff} \quad C, \eta \models \phi :$$

The Boolean cases are trivial. The case for existential quantification is as in the classical case. The case $x \heartsuit [y : \phi]$ is as follows: $Y, \eta \models x \heartsuit [y : \phi]$ iff $\zeta(\eta(y)) \models \heartsuit \llbracket \phi \rrbracket_{Y, \zeta, \eta} = \heartsuit (\llbracket \phi \rrbracket_{C, \eta} \cap Y)$ (where the equality holds by induction) iff (by ω -boundedness) $\zeta(\eta(y)) \models \heartsuit A$ for some $A \subseteq_{fin} \llbracket \phi \rrbracket_{C, \eta} \cap Y$, equivalently $\xi(\eta(y)) \models \heartsuit A$ by (5.3). The latter implies $C, \eta \models x \heartsuit [y : \phi]$ by monotonicity; conversely, $C, \eta \models x \heartsuit [y : \phi]$ implies $\xi(\eta(y)) \models \heartsuit f_{x \heartsuit [y : \phi]}(\eta)$ by construction, and $f_{x \heartsuit [y : \phi]}(\eta) \subseteq_{fin} \llbracket \phi \rrbracket_{C, \eta} \cap Y$. \square

Example 5.11. The above version of the downward Löwenheim-Skolem theorem applies to our main bounded examples (relational, graded, and positive Presburger modalities) as well as to probabilistic modalities over non-standard or zerodimensional subdistributions, respectively, which are ω -bounded but not k -bounded for any k (Examples 3.30 and 3.31).

Finally, we note that the downward Löwenheim-Skolem theorem holds also for the one-step compact case; this is in mild generalization of a corresponding result for the neighbourhood case proved already by Chang [Cha73].

Theorem 5.12. *If a Λ -structure is one-step compact, then $\text{CPL}(\Lambda, \Sigma)$ satisfies the downward Löwenheim-Skolem.*

Proof. Let \mathcal{R} denote the set of all sound one-step rules; then \mathcal{R} is one-step cut-free complete [Sch07, Proof of Theorem 18]. Let Φ be a set of coalgebraic first-order formulas in $\text{CPL}(\Lambda, \Sigma)$, and let $\mathfrak{M} = (C, \xi, V)$ be such that $\mathfrak{M} \models \Phi$. Pick Skolem functions for all formulas $\exists x. \phi$ as usual, and for every rule $R = \mathbf{A}/\mathbf{P}$ over sVar in \mathcal{R} fix a Skolem function that given an element $x \in X$ satisfying $\neg \mathbf{P}$ picks an element of X that satisfies $\neg \mathbf{A}$. More precisely: let $\sigma : \text{sVar} \rightarrow \text{CPL}(\Lambda, \Sigma)$ be a substitution for sVar , let $\psi^x \sigma$ be the formula obtained by replacing in \mathbf{P} each modal operator application $\heartsuit \mathbf{p}$ with $x \heartsuit [y : \sigma(\mathbf{p})]$ where x and y are fresh variables, and let v be a valuation such that $\mathfrak{M}, v \models \neg \psi^x \sigma$. Then there exists a y -variant v' of v such that $\mathfrak{M}, v' \models \neg \mathbf{A} \sigma$, and the Skolem function $f_{R, \sigma}$ assigns such a $v'(y)$ to $v|_{FV(\sigma)}$. As $FV(\sigma)$ is finite, $f_{R, \sigma}$ is a finitary function, so that closing a given countably infinite set $Y_0 \subseteq X$ under the Skolem functions yields a countable set $Y \subseteq X$.

The coalgebra structure ζ that we are to define on Y has to satisfy the *coherence* condition

$$\zeta(c) \models \heartsuit(\llbracket \rho \rrbracket_v^y \cap Y) \text{ iff } \xi(c) \models \heartsuit \llbracket \rho \rrbracket_v^y$$

for all $c \in Y$, all formulas ρ , and all valuations v in Y , where the second condition is by definition equivalent to $c \in \llbracket x \heartsuit [y : \rho] \rrbracket_v^x$. Once this is established, we can show as usual that (Y, ζ) is elementarily equivalent to (X, ξ) , and we are done.

To show that $\zeta(c)$ as required exists, it suffices by one-step compactness and the definition of \mathcal{R} to show that the one-step constraint implicit in the coherence condition is consistent w.r.t. \mathcal{R} . So let \mathbf{A}/\mathbf{P} be a rule over sVar in \mathcal{R} and let $\sigma : \text{sVar} \rightarrow \text{CPL}(\Lambda, \Sigma)$ such that $\xi(c) \in \llbracket \neg \mathbf{P} \rrbracket_v^y$, where $\llbracket \sigma \rrbracket_v^y$ is the $\mathcal{P}(X)$ -valuation sending $\mathbf{p} \in \text{sVar}$ to $\llbracket \sigma(\mathbf{a}) \rrbracket_v^y$ (interjection: we can indeed assume that v is the same throughout by making copies of variables). Then $f_{R, \sigma}(v|_{FV(\sigma)}) \in \llbracket \neg \phi \rrbracket_v^y$, and hence the premise of \mathbf{A}/\mathbf{P} does not hold in \mathfrak{M} when instantiated according to σ . \square

Example 5.13. Besides the plain neighbourhood case, Theorem 5.12 covers all instances of CPL defined by imposing rank-1 frame conditions on neighbourhood frames, e.g. CPL over monotone neighbourhood frames and various deontic logics.

6. PROOF THEORY

6.1. Sequent system for CPL. In §3, we have seen a complete Hilbert calculus for coalgebraic predicate logic. The present goal is a cut-free, complete sequent calculus. Our basis is the system **G1c** of [TS96] that we extend with modal rules describing the (fixed) Λ -structure. Our treatment of equality, on the other hand, is inspired by Kanger [Kan57], Degtyarev and Voronkov [DV01] and Seligman [Sel01]. In fact, the syntactic cut-elimination proof presented here is based on Seligman's ideas.

We take *sequents* to be pairs (Γ, Δ) , written $\Gamma \Rightarrow \Delta$ where $\Gamma, \Delta \subseteq \mathcal{L}$ are finite multisets. The sequent calculus for coalgebraic predicate logic contains four types of rules: the standard logical and structural rules for first-order logic, rules for equality and rules for the modal operators. The logical rules are standard as in Table 3. The formula introduced in the conclusion of a logical rule is called the *principal* formula of the rule. This applies, in particular, to the structural rules in Table 3: the formula ϕ in the conclusion is the principal one. Note that, somewhat counterintuitively, in the equality rules the formula $x = y$ in the conclusion is the *context*, i.e., the only *non-principal* formula and all the remaining ones are *principal*!

To account for the modal operators, we incorporate the one-step rules \mathcal{R} into the sequent system and write ϕ_i^j for $\sigma(p_i^j)$ as in $\text{ONESTEP}(\mathcal{R})$. Then, we transform the axiom into its sequent form as follows:

$$\frac{\Gamma_1 \sigma_{\mathbf{x}}^y \Rightarrow \Delta_1 \sigma_{\mathbf{x}}^y \quad \cdots \quad \Gamma_k \sigma_{\mathbf{x}}^y \Rightarrow \Delta_k \sigma_{\mathbf{x}}^y}{z\heartsuit_1[\mathbf{x}_1 : \phi_1], \dots, z\heartsuit_n[\mathbf{x}_n : \phi_n] \Rightarrow z\heartsuit_{n+1}[\mathbf{x}_{n+1} : \phi_{n+1}], \dots, z\heartsuit_{n+m}[\mathbf{x}_{n+m} : \phi_{n+m}]}$$

Furthermore, we add weakening contexts Σ, Θ to both the conclusion and all the premises. Finally, we obtain the desired form of $\mathcal{S}(R)$ in Table 3. The formulas $z\heartsuit_i[\mathbf{x} : \phi_i]$ are the principal formulas of $\mathcal{S}(R)$.

Example 6.1. If \mathbf{K} is the (one-step sound and one-step complete) rule set for the normal modal logic consisting of the rules

$$\frac{p \Rightarrow q_1, \dots, q_n}{\Diamond p \Rightarrow \Diamond q_1, \dots, \Diamond q_n} \mathbf{K}_n$$

for all $n \geq 0$, we obtain the following first-order version

$$\frac{\Sigma, \phi_0[y/x_0] \Rightarrow \phi_1[y/x_1], \dots, \phi_n[y/x_n], \Theta}{\Sigma, z\Diamond[x_0 : \phi_0] \Rightarrow z\Diamond[x_1 : \phi_1], \dots, z\Diamond[x_n : \phi_n], \Theta} \mathcal{S}(\mathbf{K}_n)$$

(where y is fresh in the conclusion) by the previous definition. Modal neighbourhood semantics is axiomatised by the one-step rule

$$\frac{p \Rightarrow q \quad q \Rightarrow p}{\Box p \Rightarrow \Box q} \mathbf{C}$$

which expresses that \Box is a congruential operator. The first order version of \mathbf{C} then reads

$$\frac{\Sigma, \phi_0[y/x_0] \Rightarrow \phi_1[y/x_1], \Theta \quad \Sigma, \phi_1[y/x_1] \Rightarrow \phi_0[y/x_0], \Theta}{\Sigma, z\Box[x_0 : \phi_0] \Rightarrow z\Box[x_1 : \phi_1], \Theta} \mathcal{S}(\mathbf{C})$$

(where y is fresh in the conclusion) which provides a complete and, as we are going to see below, cut-free axiomatisation of Chang's original logic.

If \mathcal{R} is a set of one-step rules, we write $\mathcal{SR} \vdash \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ can be derived using the logical and equality rules of Table 3, together with the rules $\mathcal{S}(R)$ from Table 3 for every rule $R \in \mathcal{R}$. We write $\mathcal{SRCut} \vdash \Gamma \Rightarrow \Delta$ if the *cut rule* Cut of Table 3 is used additionally. If $\mathfrak{M} = (C, \gamma, I)$ is a first-order model over a Λ -structure, we write $\mathfrak{M}, v \models \Gamma \Rightarrow \Delta$ if $\mathfrak{M}, v \models \bigwedge \Gamma \rightarrow \bigvee \Delta$ and, as usual $\mathfrak{M} \models \Gamma \Rightarrow \Delta$ if $\mathfrak{M}, v \models \Gamma \Rightarrow \Delta$ for all variable assignments v and finally $\models \Gamma \Rightarrow \Delta$ if $\mathfrak{M} \models \Gamma \Rightarrow \Delta$ for all first-order models \mathfrak{M} over the corresponding structure we keep implicit in the notation.

Proposition 6.2. *Suppose that \mathcal{R} is one-step sound. For any one-step rule $R \in \mathcal{R}$ and any model $\mathfrak{M} = (C, \gamma, I)$, $\mathcal{S}(R)$ preserves the validity on \mathfrak{M} .*

Proof. Let $R \in \mathcal{R}$ be

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \cdots \quad \Gamma_k \Rightarrow \Delta_k}{\heartsuit_1 \vec{p}_1, \dots, \heartsuit_n \vec{p}_n \Rightarrow \heartsuit_{n+1} \vec{p}_{n+1}, \dots, \heartsuit_{n+m} \vec{p}_{n+m}} .$$

Suppose that R is one-step sound and let $\mathfrak{M} = (C, \gamma, I)$ be a model. To show that $\mathcal{S}(R)$ preserves validity, assume that all of $\Sigma, \Gamma_i \sigma_{\mathbf{x}}^y \Rightarrow \Delta_i \sigma_{\mathbf{x}}^y, \Theta$ ($1 \leq i \leq k$) are valid in \mathfrak{M} . Fix any variable assignment v on C . To show that the conclusion of $\mathcal{S}(R)$ is true at M, v , assume that $M, v \models \bigwedge \Sigma$ and $M, v \not\models \bigvee \Theta$. Our goal is to show that

$$M, v \models \bigwedge_{1 \leq i \leq n} z\heartsuit_i[\mathbf{x}_i : \phi_i] \rightarrow \bigvee_{1 \leq j \leq m} z\heartsuit_{n+j}[\mathbf{x}_{n+j} : \phi_{n+j}],$$

Table 3: Sequent System of Coalgebraic Predicate Logic

Axioms	
$\frac{}{\varphi \Rightarrow \varphi} \text{Ax}$	$\frac{}{\perp \Rightarrow} \text{L}\perp \quad \frac{}{\Rightarrow x = x} \text{R} =$
Logical Rules	
$\frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \psi} \text{R} \rightarrow$	$\frac{\Gamma \Rightarrow \Delta, \phi \quad \psi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \psi, \Gamma \Rightarrow \Delta} \text{L} \rightarrow$
$\frac{\Gamma \Rightarrow \Delta, \phi[y/x]}{\Gamma \Rightarrow \Delta, \forall x. \phi} \text{R}\forall\ddagger$	$\frac{\phi[z/x], \Gamma \Rightarrow \Delta}{\forall x. \phi, \Gamma \Rightarrow \Delta} \text{L}\forall$
where \ddagger means that y is fresh in the conclusion.	
Equality Rules	
$\frac{x = y, \Gamma[x/z] \Rightarrow \Delta[x/z]}{x = y, \Gamma[y/z] \Rightarrow \Delta[y/z]} \text{L} =_1$	$\frac{x = y, \Gamma[y/z] \Rightarrow \Delta[y/z]}{x = y, \Gamma[x/z] \Rightarrow \Delta[x/z]} \text{L} =_2$
Modal Rules $\mathcal{S}(\mathcal{R})$: for every one-step rule $R \in \mathcal{R}$,	
$\frac{\Sigma, \Gamma_1 \sigma_{\vec{x}}^y \Rightarrow \Delta_1 \sigma_{\vec{x}}^y, \Theta \quad \dots \quad \Sigma, \Gamma_k \sigma_{\vec{x}}^y \Rightarrow \Delta_k \sigma_{\vec{x}}^y, \Theta}{\Sigma, z\heartsuit_1[\vec{x}_1 : \phi_1], \dots, z\heartsuit_n[\vec{x}_n : \phi_n] \Rightarrow z\heartsuit_{n+1}[\vec{x}_{n+1} : \phi_{n+1}], \dots, z\heartsuit_{n+m}[\vec{x}_{n+m} : \phi_{n+m}], \Theta} \mathcal{S}(R)$	
where	
<ul style="list-style-type: none"> y is fresh in the conclusion, $R \in \mathcal{R}$ is of the form $\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_k \Rightarrow \Delta_k}{\heartsuit_1 \vec{p}_1, \dots, \heartsuit_n \vec{p}_n \Rightarrow \heartsuit_{n+1} \vec{q}_1, \dots, \heartsuit_{n+m} \vec{q}_m} R$ where Γ_i, Δ_i are multisets of schematic variables occurring in $\vec{p}_1, \dots, \vec{p}_n, \vec{q}_1, \dots, \vec{q}_m$, $[\vec{x}_i : \phi_i] = [x_i^1 : \phi_i^1] \dots [x_i^{\text{ar}\heartsuit} : \phi_i^{\text{ar}\heartsuit}]$ is a finite sequence of comprehension formulas according to $\text{ar}\heartsuit$ and $\sigma_{\vec{x}}^y$ sends each \vec{p}_i^j to a formula $\phi_i^j[y/x_i^j]$ of \mathcal{L}. 	
Structural Rules	
$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi} \text{RW}$	$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \text{LW}$
$\frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi} \text{RC}$	$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \text{LC}$
Cut Rule (optional)	
$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{Cut}$	

i.e.,

$$\text{if } \gamma(v(z)) \in \bigcap_{1 \leq i \leq n} \llbracket \heartsuit_i \rrbracket_{C,v}(\llbracket \phi_i^1 \rrbracket_{C,v}^{x_i^1}, \dots, \llbracket \phi_i^{\text{ar}\heartsuit_i} \rrbracket_{C,v}^{x_i^{\text{ar}\heartsuit_i}}) \\ \text{then } \gamma(v(z)) \in \bigcup_{1 \leq j \leq m} \llbracket \heartsuit_{n+j} \rrbracket_{C,v}(\llbracket \phi_{n+j}^1 \rrbracket_{C,v}^{x_{n+j}^1}, \dots, \llbracket \phi_{n+j}^{\text{ar}\heartsuit_{n+j}} \rrbracket_{C,v}^{x_{n+j}^{\text{ar}\heartsuit_{n+j}}}).$$

Let us define a valuation $\tau : \text{sVar} \rightarrow \mathcal{P}(C)$ by $\tau(\vec{p}_i^j) = \llbracket \phi_i^j[y/x_i^j] \rrbracket_{C,v}^y$. To show that $C, \tau \models \bigwedge \Gamma_i \rightarrow \bigvee \Delta_i$ for all $1 \leq i \leq k$, let us fix any $c \in C$. Since y is fresh in the conclusion of $\mathcal{S}(R)$, it follows from $M, v \models \bigwedge \Sigma$ and $M, v \not\models \bigvee \Theta$ that $M, v[c/y] \models \bigwedge \Sigma$ and

$M, v[c/y] \not\models \bigvee \Theta$. Then from our assumption of the validity of all premises of $\mathcal{S}(R)$ on a pair (M, v) , we obtain $M, v[c/y] \models \Gamma_i \sigma_{\mathbf{x}}^y \Rightarrow \Delta_i \sigma_{\mathbf{x}}^y$, which implies $c \in \tau(\bigwedge \Gamma_i \rightarrow \bigvee \Delta_i)$, as desired. Since R is one-step sound, we have that $TC, \tau \models \bigwedge_{1 \leq i \leq n} \heartsuit_i \vec{p}_i \rightarrow \bigvee_{1 \leq j \leq m} \heartsuit_{n+j} \vec{p}_{n+j}$. Because $\tau(\vec{p}_i^j) = \llbracket \phi_i^j[y/x_i^j] \rrbracket_{C,v}^y = \llbracket \phi_i^j \rrbracket_{C,v}^{x_i^j}$ by freshness of y , we can conclude our desired implication above. \square

We show soundness and completeness of the sequent system \mathcal{SR} by translating into, and from, the Hilbert system \mathcal{HR} which is known to be (semantically) complete. Before showing that both systems \mathcal{HR} and \mathcal{SRCut} have the same deductive power, we note one consequence of the congruence rule provided that the rules absorb congruence. We introduce the concept of absorption in a slightly more general form which will be used later.

Definition 6.3. We say that a finite set \mathbb{S} of sequents *covers* a finite set \mathbb{S}' of sequents if each element $\Gamma \Rightarrow \Delta$ of \mathbb{S}' contains some element $\Pi \Rightarrow \Sigma$ of \mathbb{S} in the sense that $\Pi \subseteq \Gamma$ and $\Sigma \subseteq \Delta$. We write $\mathbb{S} \triangleright \mathbb{S}'$ if \mathbb{S} covers \mathbb{S}' where we identify sequents with singleton sets. A set \mathcal{R} of rules *absorbs* a rule $\Sigma_1 \Rightarrow \Theta_1, \dots, \Sigma_m \Rightarrow \Theta_m / \Sigma \Rightarrow \Theta$ if there exists a rule $R = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that

$$\{\Sigma_1 \Rightarrow \Theta_1, \dots, \Sigma_m \Rightarrow \Theta_m\} \triangleright \{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n\}$$

and $\Gamma_R \Rightarrow \Delta_R \triangleright \Sigma \Rightarrow \Theta$. A rule set *absorbs congruence* if it absorbs the rule

$$\frac{p_1 \Rightarrow q_1 \quad \dots \quad p_n \Rightarrow q_n \quad q_1 \Rightarrow p_1 \quad \dots \quad q_n \Rightarrow p_n}{\heartsuit(p_1, \dots, p_n) \Rightarrow \heartsuit(q_1, \dots, q_n)} \text{ Cong}\heartsuit$$

and it *absorbs monotonicity of \heartsuit in the i -th argument* if the rule

$$\frac{p_i \Rightarrow q_i}{\heartsuit(p_1, \dots, p_n) \Rightarrow \heartsuit(p_1, \dots, p_{i-1}, q_i, p_{i+1}, \dots, p_n)} \text{ Mon}_i$$

is absorbed.

Lemma 6.4. When \mathcal{R} absorbs congruence, $\mathcal{SR} \vdash \Gamma, \phi \Rightarrow \phi, \Delta$ for all formulas ϕ .

Proof. As \mathcal{R} absorbs congruence, the rule $\text{Cong}\heartsuit$

$$\frac{\{\Sigma, \phi_0^i[y/x_0^i] \Rightarrow \phi_1^i[y/x_1^i], \Theta \mid \Sigma, \phi_1^i[y/x_1^i] \Rightarrow \phi_0^i[y/x_0^i], \Theta \mid 1 \leq i \leq n\}}{\Sigma, z\heartsuit[\mathbf{x}_0 : \phi_0] \Rightarrow z\heartsuit[\mathbf{x}_1 : \phi_1], \Theta} \text{ Cong}\heartsuit$$

(where y is fresh in the conclusion and n is the arity of \heartsuit) is admissible in \mathcal{SR} (and \mathcal{SRCut}). This allows us to proceed by induction on the structure of ϕ , where $\text{Cong}\heartsuit$ deals with the inductive case where ϕ is of the form $x\heartsuit[\mathbf{y} : \phi]$. \square

By our equality rules, the following lemma is immediate.

Lemma 6.5. The replacement axiom $x = y, \phi[x/z] \Rightarrow \phi[y/z]$ is derivable in \mathcal{SR} .

One direction of the translation between the two proof systems can now be given as follows:

Theorem 6.6. Suppose that \mathcal{R} absorbs congruence and let $\mathcal{HR} \vdash \phi$. Then $\mathcal{SRCut} \vdash \Rightarrow \phi$.

Proof. First, we demonstrate admissibility of modus ponens in \mathcal{SRCut} by

$$\frac{\Rightarrow \phi \quad \frac{\Rightarrow \phi \rightarrow \psi \quad \phi \rightarrow \psi, \phi \Rightarrow \psi}{\phi \Rightarrow \psi} \text{ Cut}}{\Rightarrow \psi} \text{ Cut}$$

where the derivability of $\phi \rightarrow \psi, \phi \Rightarrow \psi$ is easily established by Lemma 6.4. Note that this is the only place in this proof where we need *Cut*. Hence, it suffices to show that all the axioms of \mathcal{HR} (recall Table 3) are derivable in cut-free \mathcal{SR} . All of equality axioms EN5, EN6.1 and EN6.2 are derivable by the equality axiom $R =$ and the equality rules $L =_i$. Moreover, since this is easy to show for logical but non-modal axioms, we focus on *CONG*, *ALPHA* and *ONESTEP*(\mathcal{R}). Firstly, the derivability of *CONG* follows from Lemma 6.4. Secondly, for *ALPHA* we have the following derivation:

$$\frac{\{\phi_j[y/x_j] \Rightarrow \phi_j[y/x_j] \mid j \neq i\} \quad \phi_i[y/x_i] \Rightarrow \phi_i[u/x_i][y/u] \quad \phi_i[u/x_i][y/u] \Rightarrow \phi_i[y/x_i]}{z \heartsuit [x_0 : \phi_0] \cdots [x_i : \phi_i] \cdots [x_n : \phi_n] \Rightarrow z \heartsuit [x_0 : \phi_0] \cdots [u : \phi_i[u/x_i]] \cdots [x_n : \phi_n]} \text{Cong}\heartsuit$$

where we note that *Cong* \heartsuit is admissible by the absorption of congruence. All the premises are derivable by Lemma 6.4 since u is assumed to be fresh in ϕ_i . Finally, let us move to the provability of *ONESTEP*(\mathcal{R}). Suppose that $R = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R$ is a one-step rule as in Definition 3.2. With the help of contraction rules, we note that the following are derivable rules in \mathcal{SR} : for any finite multiset Θ ,

$$\frac{\Theta, \Gamma \Rightarrow \Delta}{\bigwedge \Theta, \Gamma \Rightarrow \Delta} L\wedge \quad \frac{\Gamma \Rightarrow \Delta, \Theta}{\Gamma \Rightarrow \Delta, \bigvee \Theta} R\vee.$$

We obtain the following derivation where $N = \{1, \dots, n\}$, $M = \{n+1, \dots, n+m\}$ and π_i is an abbreviation of $(\bigwedge \Gamma_i \rightarrow \bigvee \Delta_i)\sigma$:

$$\frac{\frac{\frac{\{\pi_1[y/x] \wedge \cdots \wedge \pi_n[y/x], (\Gamma_i\sigma)[y/x] \Rightarrow (\Delta_i\sigma)[y/x] \mid 1 \leq i \leq k\}}{\{\forall x.(\pi_1 \wedge \cdots \wedge \pi_n), (\Gamma_i\sigma)[y/x] \Rightarrow (\Delta_i\sigma)[y/x] \mid 1 \leq i \leq k\}} L\vee}{\forall x.(\pi_1 \wedge \cdots \wedge \pi_n), \{x \heartsuit_i [\mathbf{x} : \phi_i] \mid i \in N\} \Rightarrow \{x \heartsuit_i [\mathbf{x} : \phi_i] \mid i \in M\}} S(R)}{\forall x.(\pi_1 \wedge \cdots \wedge \pi_n), \bigwedge \{x \heartsuit_i [\mathbf{x} : \phi_i] \mid i \in N\} \Rightarrow \bigvee \{x \heartsuit_i [\mathbf{x} : \phi_i] \mid i \in M\}} L\wedge, R\wedge$$

which shows derivability of the axiom *ONESTEP*(\mathcal{R}) as the top sequent is readily seen to be derivable in \mathcal{SR} . \square

For the converse direction, absorption of congruence is not required.

Theorem 6.7. *Suppose that $\mathcal{SRCut} \vdash \Gamma \Rightarrow \Delta$. Then $\mathcal{HR} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.*

Proof. It suffices to show that all the translations of the axioms and rules of \mathcal{SR} are derivable in \mathcal{HR} . We can easily handle the cases of the axioms and rules for logical connectives of first-order logic. The provability of the translation of $L =_i$ follows from the provability of $x = y \rightarrow (\phi[x/w] \rightarrow \phi[y/w])$. As for $\heartsuit \in \Lambda$, the provability of the translation of $S(R)$ follows from *ONESTEP*(\mathcal{R}) and *ALPHA*. \square

As a corollary, we obtain (for the time being, in a calculus with cut) both soundness and completeness of the sequent calculus.

Corollary 6.8. *Suppose that \mathcal{R} is one-step sound and strongly one-step complete. Then $\mathcal{SRCut} \vdash \Gamma \Rightarrow \Delta$ iff $\models \Gamma \Rightarrow \Delta$.*

Proof. By Theorems 6.6 and 6.7 in conjunction with soundness and completeness of \mathcal{HR} (Theorem 3.15). The absorption of congruence was shown in [PS10, Proposition 5.12]. \square

A paradigm example of a set of rules satisfying the assumptions of Corollary 6.8 is \mathbf{C} and its CPL translation $\mathcal{S}(\mathbf{C})$ from Example 6.1 above.

As we have seen in §3, the assumption of *strongly* one-step complete rule sets limits available examples to “essentially neighbourhood-like” ones. This is why we also gave a complete Hilbert-style axiomatisation also for *bounded* operators (recall Definition 3.8).

Note that k -boundedness of i -th argument of Definition 3.8 implies in particular that \heartsuit is monotonic in the i -th argument. Examples of bounded modalities include the standard \Diamond of relational modal logic interpreted over Kripke frames, graded modalities over multigraphs and we refer to [SP10b] for more examples. In the Hilbert-calculus, boundedness was reflected syntactically by the axiom

$$\text{BDPL}_{k,i} \forall \vec{y}. (x \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n] \leftrightarrow \exists z_1 \dots z_k. (x \heartsuit [y_1 : \phi_1] \dots [y_{i-1} : \phi_{i-1}] \\ [y_i : y_i = z_1 \vee \dots \vee y_i = z_k] [y_{i+1} : \phi_{i+1}] \dots [y_n : \phi_n] \wedge \bigwedge_{j \leq k} \phi_i[y_i/z_j]))$$

where each z_i is fresh for all the y_i s and ϕ_i s. The derivability predicate induced by extending the Hilbert calculus \mathcal{HR} by the boundedness axiom above gives completeness under weaker conditions.

Definition 6.9. We write $\mathcal{BHR} \vdash \phi$ if ϕ is derivable in \mathcal{HR} where additionally $\text{BDPL}_{k,i}$ is used for every operator that is k -bounded in the i -th argument.

Strictly speaking, the derivability predicate \mathcal{BHR} should include information about precisely which operators are assumed to be k -bounded in the i -th argument, but this will always be clear from the context. In the presence of boundedness, completeness of the Hilbert-calculus has been established under weaker conditions (see Theorem 3.15).

We can reflect boundedness in the sequent calculus by adding a paste rule, similar in spirit to the paste rule of hybrid logic [BdRV01, §7] which was generalised to a coalgebraic setting in [SP10b]. In a sequent setting, this rule takes the form

$$\frac{\Gamma, x \heartsuit [x_1 : \phi_1] \dots [x_{i-1} : \phi_{i-1}] [y : \bigvee_{1 \leq j \leq k} y = z_j] [x_{i+1} : \phi_{i+1}] \dots [x_n : \phi_n], \\ \phi[z_1/y], \dots, \phi[z_k/y] \Rightarrow \Delta \quad z_1, \dots, z_k \text{ fresh}}{\Gamma, x \heartsuit [x_1 : \phi_1] \dots [x_{i-1} : \phi_{i-1}] [y : \phi] [x_{i+1} : \phi_{i+1}] \dots [x_n : \phi_n] \Rightarrow \Delta} \text{Paste}_i^k$$

where z_1, \dots, z_k are pairwise distinct fresh variables. Additional use of the above paste-rule in the system \mathcal{SR} is denoted by \mathcal{BSR} , that is, we write $\mathcal{BSR} \vdash \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is derivable in \mathcal{SR} where Paste_i^k may additionally be applied for every modality that is k -bounded in the i -th argument.

When \mathcal{R} absorbs congruence and monotonicity of all operators that are k -bounded in the i -th argument, we note that Lemmas 6.4 and 6.5 hold also for \mathcal{BSR} .

Theorem 6.10. *Suppose that \mathcal{R} absorbs congruence and monotonicity in the i -th argument of every operator that is k -bounded in the i -th argument. Then $\mathcal{BHR} \vdash \phi$ implies that $\mathcal{BSRCut} \vdash \phi$.*

Proof. First of all, if \mathcal{R} absorbs monotonicity in the i -th argument of $\heartsuit \in \Lambda$, the rule

$$\frac{\Sigma, \phi_i[y/x_i] \Rightarrow \psi[y/x], \Theta}{\Sigma, z \heartsuit [x : \phi] \Rightarrow z \heartsuit [x_1 : \phi_1] \dots [x_{i-1} : \phi_{i-1}] [x_i : \psi] [x_{i+1} : \phi_{i+1}] \dots [x_n : \phi_n], \Theta} \text{Mon}_i$$

(where y is fresh in the conclusion) is admissible in \mathcal{BSR} (and \mathcal{BSRCut}). Almost all the arguments are the same as the proof of Theorem 6.6, except that we need to show the provability of BDPL by **Paste** (note that the only place we need the cut rule is the derivability of Modus Ponens). More precisely, we can show the left-to-right implication of BDPL by means of \mathbf{Paste}_i^k and \mathbf{Mon}_i gives the reverse direction. For example, when \heartsuit is unary and 1-bounded, the derivability of the right-to-left direction of BDPL is demonstrated as follows.

$$\frac{\frac{\frac{v = w, \phi[w/y] \Rightarrow \phi[w/y]}{v = w, \phi[w/y] \Rightarrow \phi[v/y]} \mathbf{L} =_2}{\frac{x\heartsuit[y : y = w], \phi[w/y] \Rightarrow x\heartsuit[y : \phi]}{x\heartsuit[y : y = w] \wedge \phi[w/y] \Rightarrow x\heartsuit[y : \phi]} \mathbf{Mon}}{\frac{x\heartsuit[y : y = w] \wedge \phi[w/y] \Rightarrow x\heartsuit[y : \phi]}{\exists z.(x\heartsuit[y : y = z] \wedge \phi[z/y]) \Rightarrow x\heartsuit[y : \phi]} \mathbf{L}\wedge} \mathbf{L}\exists,$$

where the top sequent is the replacement axiom, which is derivable by Lemma 6.5. \square

The reverse direction of Theorem 6.10 is established analogously to Theorem 6.7 and again absorption properties are not needed.

Theorem 6.11. $\mathcal{BSRCut} \vdash \Gamma \Rightarrow \Delta$ only if $\mathcal{BHR} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Proof. The only difference from the proof of Theorem 6.6 is to need to care about the translation of **Paste**. However, we can easily establish this by the axiom BDLP. \square

As in the non-bounded case we obtain semantic soundness and completeness, but under weaker coherence conditions.

Corollary 6.12. Suppose that \mathcal{R} is one-step sound and strongly finitary one-step complete. Then $\mathcal{BSRCut} \vdash \Gamma \Rightarrow \Delta$ iff $\models \Gamma \Rightarrow \Delta$.

Proof. By Theorems 6.10 and 6.11 in conjunction with soundness and completeness of \mathcal{BHR} (Theorem 3.15). Note that absorption of congruence and monotonicity follows from (strong, finitary) one-step completeness as in [PS10, Proposition 5.12]. \square

A canonical example of a rule set satisfying the assumptions of the above corollary can be obtained by taking \mathbf{K} of Example 6.1 and extending it with \mathbf{Paste}_i^k for $i = k = n = 1$.

6.2. Admissibility of Cut. When we try to prove the admissibility of Cut in first-order logic (or \mathcal{SR}), we encounter difficulties with the rules of contraction. That is, the following derivation:

$$\mathcal{D} = \frac{\frac{\frac{\mathcal{D}'}{\Gamma \Rightarrow \Delta, \phi, \phi} \mathbf{RC}}{\Gamma \Rightarrow \Delta, \phi} \quad \frac{\mathcal{D}''}{\phi, \Sigma \Rightarrow \Theta} \mathbf{Cut}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \mathbf{Cut},$$

may be transformed into:

$$\frac{\frac{\frac{\mathcal{D}'}{\Gamma \Rightarrow \Delta, \phi, \phi} \quad \frac{\mathcal{D}''}{\phi, \Sigma \Rightarrow \Theta}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi} \mathbf{Cut} \quad \frac{\mathcal{D}''}{\phi, \Sigma \Rightarrow \Theta}}{\frac{\Gamma, \Sigma, \Sigma \Rightarrow \Delta, \Theta, \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \mathbf{LC, RC}} \mathbf{Cut},$$

but this derivation does not provide us with a reduction in terms of the number of sequents above the application of Cut in \mathcal{D} . This is why Gentzen introduced the following generalized form of Cut:

$$\frac{\Gamma \Rightarrow \Delta, \phi^m \quad \phi^n, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{Mcut}$$

where $n, m \geq 1$ and ϕ^k stands for k copies of ϕ and “Mcut” is a shorthand of “multi-cut” (sometimes also called “mix”). Since Cut is a special case of the new rule of Mcut, it suffices for us to prove the admissibility of Mcut in a given sequent system to obtain the admissibility of Cut in the system.

Moreover, we note that *a priori* we cannot expect that the elimination holds for the application of Mcut between two instances of modal rules: the set \mathcal{R} of one-step rules can possibly consist of a single rule, and an application of Mcut between this rule and itself may not be derivable. We therefore need to impose an additional requirement to deal with this case.

Definition 6.13. Let \mathbb{S} be a finite set of sequents. The set of all sequents that can be derived from premises in \mathbb{S} using (only) *one application* of Mcut is denoted by $\text{MCut}(\mathbb{S})$. A rule set \mathcal{R} *absorbs multicut*, if for all pairs (R_1, R_2) of rules in \mathcal{R} :

$$\frac{\Gamma_{11} \Rightarrow \Delta_{11} \quad \cdots \quad \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}}{\Gamma_1 \Rightarrow \Delta_1, (\heartsuit \vec{p})^m} R_1 \quad \frac{\Gamma_{21} \Rightarrow \Delta_{21} \quad \cdots \quad \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}}{(\heartsuit \vec{p})^n, \Gamma_2 \Rightarrow \Delta_2} R_2$$

there is a rule $R = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that:

$\text{MCut}(\Gamma_{11} \Rightarrow \Delta_{11}, \dots, \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}, \Gamma_{21} \Rightarrow \Delta_{21}, \dots, \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}) \triangleright \{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k\}$
and $\Gamma_R \Rightarrow \Delta_R \triangleright \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

Lemma 6.14. *If $\mathcal{SR} \vdash \Gamma \Rightarrow \Delta$ and y is fresh in Γ and Δ , then $\text{If } \mathcal{SR} \vdash \Gamma[y/x] \Rightarrow \Delta[y/x]$ with the same height of derivation.*

Lemma 6.15 (Hauptsatz). *Let \mathcal{D} be a derivation in the system \mathcal{SR} extended with Mcut in the following form:*

$$\frac{\frac{\mathcal{D}_L}{\Gamma \Rightarrow \Delta, \phi^m} \quad \frac{\mathcal{D}_R}{\phi^n, \Sigma \Rightarrow \Theta}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{Mcut},$$

where \mathcal{D}_L and \mathcal{D}_R contain no application of (Mcut), the last rule of \mathcal{D} is the only application of (Mcut) in \mathcal{D} . Then $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ is derivable in \mathcal{SR} .

Proof. First of all, we introduce some terminology used only in this proof. Let \mathcal{D} be the derivation in question. We say that ϕ is a *cut formula* of \mathcal{D} , and we define the complexity $c(\mathcal{D})$ as the complexity of the cut formula ϕ , i.e., the length or the number of connectives including the logical and modal connectives. Moreover, we define $w(\mathcal{D})$ as the total number of sequents in \mathcal{D}_L and \mathcal{D}_R . Our proof of the statement of the claim is shown by the double induction on $(c(\mathcal{D}), w(\mathcal{D}))$ (note that $c(\mathcal{D}) \geq 0$ and $w(\mathcal{D}) \geq 2$). Let us denote the last applied rule (or axiom, possibly) of a derivation \mathcal{E} by $\text{rule}(\mathcal{E})$. We divide our argument into the following (exhaustive) cases:

- (1) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is an axiom.
- (2) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is a structural rule.
- (3) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is a logical rule or a modal rule and the cut formula is not principal in the rule.

- (4) Both $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ are logical rules for the same logical connective and the cut formula is principal in each of the rules.
- (5) Both $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ are modal rules and the cut formula is principal in each of the rules.
- (6) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is an equality rule.

Let us check each case one by one.

- (1) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is an axiom: We have four cases since it is impossible that $\text{rule}(\mathcal{D}_L)$ is $\text{L}\perp$ or $\text{rule}(\mathcal{D}_L)$ is $\text{R} =$. Firstly, when $\text{rule}(\mathcal{D}_L)$ is Ax , let the derivation be

$$\frac{\frac{\overline{\phi \Rightarrow \phi} \text{ Ax} \quad \phi^n, \Sigma \Rightarrow \Theta}{\phi, \Sigma \Rightarrow \Theta} \mathcal{D}_R}{\phi, \Sigma \Rightarrow \Theta} \text{Mcut}.$$

When $n = 1$, we already obtain the derivability of $\phi, \Sigma \Rightarrow \Theta$ in \mathcal{SR} . When $n \geq 2$, $\phi, \Sigma \Rightarrow \Theta$ is obtained from $\phi^n, \Sigma \Rightarrow \Theta$ by finitely many applications of RC .

Secondly, when $\text{rule}(\mathcal{D}_R)$ is Ax , the argument is similar to the previous case where $\text{rule}(\mathcal{D}_L)$ is Ax .

Thirdly, when $\text{rule}(\mathcal{D}_L)$ is $\text{R} =$, we need to look at what is the last rule $\text{rule}(\mathcal{D}_R)$ where \mathcal{D} is of the following form:

$$\frac{\frac{\overline{\Rightarrow x = x} \text{ R} = \quad (x = x)^n, \Sigma \Rightarrow \Theta}{\Sigma \Rightarrow \Theta} \mathcal{D}_R}{\Sigma \Rightarrow \Theta} \text{Mcut}.$$

If $\text{rule}(\mathcal{D}_R)$ is an axiom, then it should be Ax and we have already checked such case in our second case of this item. Otherwise, $\text{rule}(\mathcal{D}_R)$ is a structural rule, a logical rule, a modal rule or an equality rule. These cases will be discussed below (especially (2), (3) and (6)), so we leave them out for now.

Fourthly, when $\text{rule}(\mathcal{D}_R)$ is $\text{L}\perp$, then we need to look at the last rule $\text{rule}(\mathcal{D}_L)$, where \mathcal{D} is

$$\frac{\frac{\Gamma \Rightarrow \Delta, \perp^m \quad \overline{\perp \Rightarrow} \text{L}\perp}{\Gamma \Rightarrow \Delta} \mathcal{D}_L}{\Gamma \Rightarrow \Delta} \text{Mcut}.$$

If $\text{rule}(\mathcal{D}_L)$ is an axiom, it should be Ax and we have already checked such case in the first case of this item. Otherwise, $\text{rule}(\mathcal{D}_L)$ must be a structural rule, a logical rule, an modal rule or an equality rule. Again these cases will be discussed below (especially (2), (3) and (6)), so we leave them out for now.

- (2) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is a structural rule: all arguments for this case are standard, so we deal only with the case where $\text{rule}(\mathcal{D}_L)$ is RC , i.e., \mathcal{D} is of the following form:

$$\frac{\frac{\frac{\Gamma \Rightarrow \Theta, \phi^{m+1}}{\Gamma \Rightarrow \Delta, \phi^m} \mathcal{D}'_L \quad \phi^n, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{RC} \quad \mathcal{D}_R}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{Mcut},$$

since multicut plays an essential role. This derivation is transformed into:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi^{m+1}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \mathcal{D}'_L \quad \phi^n, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \mathcal{D}_R \text{Mcut}$$

where the application of (*Mcut*) is eliminable since the complexity of the derivation is the same as $c(\mathcal{D})$ and the weight of the derivation is smaller than $w(\mathcal{D})$.

- (3) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is a logical rule or a modal rule and the cut formula is not principal in the rule: Our argument for logical rules are standard, so we focus on the case where one of the rules is a modal rule $\mathcal{S}(R)$. Let $\text{rule}(\mathcal{D}_L)$ be $\mathcal{S}(R)$. Then our derivation \mathcal{D} is of the following form:

$$\frac{\frac{\Sigma', (\Gamma_1 \sigma)[y/x] \Rightarrow (\Delta_1 \sigma)[y/x], \Theta', \phi^m \quad \dots \quad \Sigma', (\Gamma_k \sigma)[y/x] \Rightarrow (\Delta_k \sigma)[y/x], \Theta', \phi^m}{\Sigma', z\heartsuit_1[\mathbf{x}:\phi_1], \dots, z\heartsuit_n[\mathbf{x}:\phi_n] \Rightarrow} \mathcal{S}(R)^\dagger \quad \frac{z\heartsuit_{n+1}[\mathbf{x}:\phi_{n+1}], \dots, z\heartsuit_{n+m}[\mathbf{x}:\phi_{n+m}], \Theta', \phi^m \quad \phi^n, \Sigma \Rightarrow \Theta}{\Sigma, \Sigma', z\heartsuit_1[\mathbf{x}:\phi_1], \dots, z\heartsuit_n[\mathbf{x}:\phi_n] \Rightarrow} \mathcal{D}_R}{z\heartsuit_{n+1}[\mathbf{x}:\phi_{n+1}], \dots, z\heartsuit_{n+m}[\mathbf{x}:\phi_{n+m}], \Theta', \Theta} \text{Mcut}$$

For each \mathcal{D}_{L_i} , we apply height-preserving substitution $[z/y]$ for a fresh variable z in the conclusion of \mathcal{D} and we obtain the following derivation:

$$\frac{\Sigma', (\Gamma_i \sigma)[z/x] \Rightarrow (\Delta_i \sigma)[z/x], \Theta', \phi^m \quad \phi^n, \Sigma \Rightarrow \Theta}{\Sigma, \Sigma', (\Gamma_i \sigma)[z/x] \Rightarrow (\Delta_i \sigma)[z/x], \Theta', \Theta} \mathcal{D}_{L_i}[z/y] \quad \mathcal{D}_R \quad \text{Mcut}.$$

We can eliminate the last application of *Mcut* since the complexity of the derivation is the same as $c(\mathcal{D})$ and the weight of the derivation is smaller than $w(\mathcal{D})$. Finally we apply the same rule $\mathcal{S}(R)$ to obtain the desired conclusion. When $\text{rule}(\mathcal{D}_R)$ be $\mathcal{S}(R)$, the argument is similar to the case just discussed.

- (4) Both $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ are logical rules for the same logical connective and the cut formula is principal in each of the rules: We have two cases, i.e., two cases where the cut formula is of the form $\phi \rightarrow \psi$ or of the form $\forall x.\phi$. Here we only deal with the case where the cut formula is of the form $\forall x.\phi$. Then the derivation \mathcal{D} is of the following form:

$$\frac{\frac{\Gamma \Rightarrow \Delta, (\forall x.\phi)^{m-1}, \phi[y/x]}{\Gamma \Rightarrow \Delta, (\forall x.\phi)^m} \mathcal{D}'_L \quad \frac{\phi[z/x], (\forall x.\phi)^{n-1}, \Sigma \Rightarrow \Theta}{(\forall x.\phi)^n, \Sigma \Rightarrow \Theta} \mathcal{D}'_R}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{R}\forall^\dagger \quad \text{L}\forall \quad \text{Mcut}.$$

With the help of our height-preserving substitution, we can consider a multicut between \mathcal{D}'_L and \mathcal{D}_R :

$$\frac{\Gamma \Rightarrow \Delta, (\forall x.\phi)^{m-1}, \phi[z/x] \quad (\forall x.\phi)^n, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x]} \mathcal{D}'_L[z/y] \quad \mathcal{D}_R \quad \text{Mcut},$$

and then by induction hypothesis (the complexity of this derivation is the same as \mathcal{D} but the weight is smaller than the original \mathcal{D}) we now know that $\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x]$ is derivable in \mathcal{SR} without multicut by a derivation \mathcal{E}_1 . Let us also consider a multicut between \mathcal{D}_L and \mathcal{D}'_R :

$$\frac{\Gamma \Rightarrow \Delta, (\forall x.\phi)^m \quad \phi[z/x], (\forall x.\phi)^{n-1}, \Sigma \Rightarrow \Theta}{\phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta} \mathcal{D}_L \quad \mathcal{D}'_R \quad \text{Mcut},$$

and then by induction hypothesis (the complexity of this derivation is the same as \mathcal{D} but the weight is smaller than the original \mathcal{D}) we now know that $\phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta$ is derivable in \mathcal{SR} without multicut by a derivation \mathcal{E}_2 . Now let us take a cut between \mathcal{E}_1 and \mathcal{E}_2 :

$$\frac{\frac{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x] \quad \phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta}{\Gamma, \Gamma, \Sigma, \Sigma \Rightarrow \Delta, \Delta, \Theta, \Theta} \text{Mcut}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x] \quad \phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta}$$

and the conclusion of this derivation is derivable in \mathcal{SR} without multicut by induction hypothesis because the complexity of this derivation (i.e., the length of $\phi[z/x]$) is strictly smaller than $c(\mathcal{D})$. Finally, finitely many applications of contraction rules enables us to obtain the derivability of $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ in \mathcal{SR} , as desired.

- (5) Both $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ are modal rules and the cut formula is principal in each of the rules: Let $\text{rule}(\mathcal{D}_L) = \mathcal{S}(R_1)$ and $\text{rule}(\mathcal{D}_R) = \mathcal{S}(R_2)$ where we can assume:

$$R_1 = \frac{\Gamma_{11} \Rightarrow \Delta_{11} \quad \cdots \quad \Gamma_{1k} \Rightarrow \Delta_{1k}}{\heartsuit_1 \vec{p}_1, \dots, \heartsuit_a \vec{p}_a \Rightarrow \heartsuit_{a+1} \vec{q}_1, \dots, \heartsuit_{a+b} \vec{q}_b, (\heartsuit \vec{p})^n},$$

$$R_2 = \frac{\Gamma_{21} \Rightarrow \Delta_{21} \quad \cdots \quad \Gamma_{2l} \Rightarrow \Delta_{2l}}{(\heartsuit \vec{p})^m, \spadesuit_1 \vec{p}'_1, \dots, \spadesuit_c \vec{p}'_c \Rightarrow \spadesuit_{c+1} \vec{q}'_1, \dots, \spadesuit_{c+d} \vec{q}'_d},$$

because the cut formula is principal in both rules. In what follows, we assume that all of $\vec{p}_i, \vec{q}_j, \vec{p}'_i, \vec{q}'_j$ are distinct. So \mathcal{D}_L is of the following form:

$$\frac{\frac{\mathcal{D}'_{L_1} \quad \cdots \quad \mathcal{D}'_{L_k}}{\Sigma_1, \Gamma_{11} \sigma_{\vec{x}}^{y_1} \Rightarrow \Delta_{11} \sigma_{\vec{x}}^{y_1}, \Theta_1 \quad \cdots \quad \Sigma_1, \Gamma_{1k} \sigma_{\vec{x}}^{y_1} \Rightarrow \Delta_{1k} \sigma_{\vec{x}}^{y_1}, \Theta_1} \mathcal{S}(R_1)}{\Sigma_1, z \heartsuit_1 [\vec{x}_1 : \phi_1], \dots, z \heartsuit_a [\vec{x}_a : \phi_a] \Rightarrow z \heartsuit_{a+1} [\vec{x}_{a+1} : \phi_{a+1}], \dots, z \heartsuit_{a+b} [\vec{x}_{a+b} : \phi_{a+b}], (z \heartsuit [\vec{x} : \phi])^m, \Theta_1}$$

and \mathcal{D}_R is of the following form:

$$\frac{\frac{\mathcal{D}'_{R_1} \quad \cdots \quad \mathcal{D}'_{R_l}}{\Sigma_2, \Gamma_{21} \tau_{\vec{x}}^{y_2} \Rightarrow \Delta_{21} \tau_{\vec{x}}^{y_2}, \Theta_2 \quad \cdots \quad \Sigma_2, \Gamma_{2l} \tau_{\vec{x}}^{y_2} \Rightarrow \Delta_{2l} \tau_{\vec{x}}^{y_2}, \Theta_2} \mathcal{S}(R_2)}{\Sigma_2, (z \heartsuit [\vec{x} : \phi])^n, z \spadesuit_1 [\vec{x}_1 : \psi_1], \dots, z \spadesuit_c [\vec{x}_c : \psi_c] \Rightarrow z \spadesuit_{c+1} [\vec{x}_{c+1} : \psi_{c+1}], \dots, z \spadesuit_{c+d} [\vec{x}_{c+d} : \psi_{c+d}], \Theta_2}.$$

We also note that the conclusion of \mathcal{D} is:

$$\Sigma_1, \Sigma_2, \{z \heartsuit_i [\vec{x}_i : \phi_i]\}_{1 \leq i \leq a}, \{z \spadesuit_j [\vec{x}_j : \psi_j]\}_{1 \leq j \leq c} \Rightarrow \{z \heartsuit_{a+i} [\vec{x}_{a+i} : \phi_{a+i}]\}_{1 \leq i \leq b}, \{z \spadesuit_{c+j} [\vec{x}_{c+j} : \psi_{c+j}]\}_{1 \leq j \leq d}, \Theta_1, \Theta_2.$$

Let y be a fresh variable not occurring in this conclusion. By height-preserving substitution, we can obtain derivations $\mathcal{D}'_{L_i}[z/y_1]$ and $\mathcal{D}'_{R_j}[z/y_2]$ ($1 \leq i \leq k$ and $1 \leq j \leq l$). Since \mathcal{R} absorbs multicut, we can find a rule $R = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_e \Rightarrow \Delta_e / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that

- (*) $\text{MCut}(\{\Gamma_{1i} \Rightarrow \Delta_{1i}\}_{1 \leq i \leq k}, \{\Gamma_{2j} \Rightarrow \Delta_{2j}\}_{1 \leq j \leq l}) \triangleright \{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_e \Rightarrow \Delta_e\}$ and
 (**) $\Gamma_R \Rightarrow \Delta_R \triangleright \{\heartsuit_i \vec{p}_i\}_{1 \leq i \leq a}, \{\spadesuit_j \vec{p}'_j\}_{1 \leq j \leq c} \Rightarrow \{\heartsuit_{a+i} \vec{q}_i\}_{1 \leq i \leq b}, \{\spadesuit_{c+j} \vec{q}'_j\}_{1 \leq j \leq d}.$

By the clause $(*_1)$ and our derivations $\mathcal{D}'_{L_i}[z/y_1]$, $\mathcal{D}'_{R_j}[z/y_2]$, we now use the induction hypothesis (the complexity is the same but the weight becomes smaller than that of \mathcal{D}) and weakening rules to obtain the derivability in \mathcal{SR} (without multicut) of

$$\Gamma_i \sigma \Rightarrow \Delta_i \sigma \quad (1 \leq i \leq e)$$

where σ is a substitution which is the union of $\sigma_x^{y_1}$ and $\tau_x^{y_2}$. It follows from the rule $\mathcal{S}(R)$, the clause $(*_2)$ and weakening rules that the conclusion of \mathcal{D} is derivable in \mathcal{SR} without multicut, as desired.

- (6) One of $\text{rule}(\mathcal{D}_L)$ and $\text{rule}(\mathcal{D}_R)$ is an equality rule: There are three cases that we need to consider. In the first case, $\text{rule}(\mathcal{D}_R)$ is $L =_i$ where at least one occurrence of the cut formulas is not principal in $L =_i$ and so the cut formula is of the form $x = y$. In the second case, $\text{rule}(\mathcal{D}_L)$ is $L =_i$ but all occurrences of the cut formula are principal. In the third case, $\text{rule}(\mathcal{D}_R)$ is $L =_i$ and all occurrences of the cut formula are principal. Since our argument for the third case is almost similar to the one for the second case, we focus on the first and the second cases in what follows.

Firstly, consider the case when $\text{rule}(\mathcal{D}_R)$ is $L =_i$ and at least one occurrence of the cut formulas is not principal in $L =_i$. Without loss of generality, we assume that $i = 1$. Then our derivation \mathcal{D} is of the following form:

$$\frac{\frac{\mathcal{D}_L}{\Gamma \Rightarrow \Delta, (x = y)^m} \quad \frac{\frac{\mathcal{D}'_R}{x = y, \phi'_1[x/w], \dots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}{x = y, \phi'_1[y/w], \dots, \phi'_{n-1}[y/w], \Sigma'[y/w] \Rightarrow \Theta'[y/w]} L =_1}{\Gamma, \Sigma \Rightarrow \Delta, \Theta} \text{Mcut}$$

where $\Sigma'[y/w] = \Sigma$, $\Theta'[y/w] = \Theta$, $\phi'_i[y/w]$ is $x = y$ and so $x = y, \phi'_1[y/w], \dots, \phi'_{n-1}[y/w]$ is the same as $(x = y)^n$. In this case we need to check what is the last rule $\text{rule}(\mathcal{D}_L)$. If $\text{rule}(\mathcal{D}_L)$ is Ax (it cannot be $L \perp$), or a structural rule, or a logical or modal rule, we can use the same argument in the items (1), (2), (3). If $\text{rule}(\mathcal{D}_L)$ is an equality rule $L =_i$, then our argument is the same as in the second case below. The remaining case is $\text{rule}(\mathcal{D}_L)$ is an axiom $R =$. Then our derivation above \mathcal{D} has the following form:

$$\frac{\frac{\frac{\Rightarrow x = x}{R =} \quad \frac{\mathcal{D}'_R}{x = x, \phi'_1[x/w], \dots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}{x = x, \phi'_1[x/w], \dots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]} L =_1}{\Sigma \Rightarrow \Theta} \text{Mcut}$$

Then this derivation is transformed into:

$$\frac{\frac{\Rightarrow x = x}{R =} \quad \frac{\mathcal{D}'_R}{x = x, \phi'_1[x/w], \dots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}{\Sigma \Rightarrow \Theta} \text{Mcut}$$

and this last application of multicut is eliminable since the complexity is the same as that of \mathcal{D} but the weight becomes smaller.

Secondly, let $\text{rule}(\mathcal{D}_L)$ be $L =_i$ and assume that all occurrences of the cut formula are principal. In this case the derivation \mathcal{D} is of the following form:

$$\frac{\frac{x = y, \Gamma''[x/w] \Rightarrow \Delta''[x/w], \phi'_1[x/w], \dots, \phi'_m[x/w]}{x = y, \Gamma''[y/w] \Rightarrow \Delta''[y/w], \phi'_1[y/w], \dots, \phi'_m[y/w]} \mathcal{D}'_L \quad \phi^n, \Sigma \Rightarrow \Theta \quad \mathcal{D}_R}{x = y, \Gamma', \Sigma \Rightarrow \Delta, \Theta} L =_1 \quad \text{Mcut}$$

where $\Gamma''[y/w] = \Gamma'$, $\Delta''[y/w] = \Delta$ and $\phi'_i[y/w] = \phi$ ($1 \leq i \leq m$). Before transforming this derivation into a multicut-free derivation, we remark that $\phi'_i[x/w][y/x] = \phi'_i[y/w][y/x] = \phi[y/x]$, $\Gamma''[x/w][y/x] = \Gamma''[y/w][y/x] = \Gamma'[y/x]$, $\Delta'[x/w][y/x] = \Delta'[y/w][y/x] = \Delta[y/x]$. With the help of this remark, the derivation \mathcal{D} is transformed into:

$$\frac{\frac{\frac{y = y, \Gamma'[y/x] \Rightarrow \Delta[y/x], (\phi[y/x])^m}{y = y, \Gamma'[y/x], \Sigma[y/x] \Rightarrow \Delta[y/x], \Theta[y/x]} \mathcal{D}'_L[y/x] \quad (\phi[y/x])^n, \Sigma[y/x] \Rightarrow \Theta[y/x] \quad \mathcal{D}_R[y/x]}{y = y, \Gamma'[y/x], \Sigma[y/x] \Rightarrow \Delta[y/x], \Theta[y/x]} \text{Mcut}$$

$$\frac{y = y, \Gamma'[y/x], \Sigma[y/x] \Rightarrow \Delta[y/x], \Theta[y/x]}{x = y, y = y, \Gamma'[y/x], \Sigma[y/x] \Rightarrow \Delta[y/x], \Theta[y/x]} \text{RW}$$

$$\frac{x = y, x = y, \Gamma', \Sigma \Rightarrow \Delta, \Theta}{x = y, \Gamma', \Sigma \Rightarrow \Delta, \Theta} L =_2 \quad \text{LC}$$

where we note that the first application of multicut is eliminable since the complexity is the same as that of \mathcal{D} but the weight is smaller than that of \mathcal{D} by height-preserving substitution $[y/x]$. □

Theorem 6.16 (Cut Elimination). *Suppose that \mathcal{R} absorbs multicut. Then the rule Mcut is admissible in \mathcal{SR} . Therefore, Cut is also admissible in \mathcal{SR} .*

Proof. Suppose that a sequent is derivable in the system \mathcal{SR} extended with Cut. Let \mathcal{E} be such a derivation. Then we focus on one of the topmost applications of Cut to show that such application of Cut is eliminable, i.e., we show that the derivation whose last applied rule is such multicut can be replaced with a multicut-free derivation of \mathcal{SR} . This is done using Lemma 6.15. Once we eliminate one of the topmost applications of Cut, we repeat the same argument for the remaining topmost applications with the help of Lemma 6.15 to get rid of all applications of Cut in the original derivation \mathcal{E} . □

In what follows, we introduce the notion of *absorption of contraction and cut* and show that jointly they provide a sufficient condition of absorption of multicut.

Definition 6.17. Let \mathbb{S} be a finite set of sequents. The set of sequents that can be derived from premises \mathbb{S} using (only) the *contraction rules* is denoted by $\text{Con}(\mathbb{S})$. Similarly, the set of all sequents that can be derived from premises in \mathbb{S} using (only) *one application* of the *cut rule* is denoted by $\text{Cut}(\mathbb{S})$. A rule set \mathcal{R} *absorbs contraction* if, for all rules $R = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ and all $\Gamma' \Rightarrow \Delta' \in \text{Con}(\Gamma_R \Rightarrow \Delta_R)$ there exists a rule $S = \Sigma_1 \Rightarrow \Theta_1, \dots, \Sigma_l \Rightarrow \Theta_l / \Gamma_S \Rightarrow \Delta_S \in \mathcal{R}$ such that

$$\text{Con}(\{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k\}) \triangleright \{\Sigma_1 \Rightarrow \Theta_1, \dots, \Sigma_l \Rightarrow \Theta_l\}$$

and $\Gamma_S \Rightarrow \Delta_S \triangleright \Gamma' \Rightarrow \Delta'$. A rule set \mathcal{R} *absorbs multicut*, if for all pairs (R_1, R_2) of rules in \mathcal{R} :

$$\frac{\Gamma_{11} \Rightarrow \Delta_{11} \quad \cdots \quad \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}}{\Gamma_1 \Rightarrow \Delta_1, \heartsuit \vec{p}} R_1 \quad \frac{\Gamma_{21} \Rightarrow \Delta_{21} \quad \cdots \quad \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}}{\heartsuit \vec{p}, \Gamma_2 \Rightarrow \Delta_2} R_2$$

there is a rule $R = \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that:

$\text{Cut}(\Gamma_{11} \Rightarrow \Delta_{11}, \dots, \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}, \Gamma_{21} \Rightarrow \Delta_{21}, \dots, \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}) \triangleright \{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k\}$
and $\Gamma_R \Rightarrow \Delta_R \triangleright \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

Informally, absorption of cut and contraction of a rule set allows us to replace an application of cut or contraction to the conclusions of rules in \mathcal{R} by a possibly different rule with possibly weaker premises and stronger conclusion. While these definitions are purely syntactic, a semantic characterisation has been given in [PS10] in terms of *one-step cut-free completeness*. For many Λ -structures including those for the modal logic K and the logic of (monotone) neighbourhood frames, one-step cut-free complete rule sets are known. In particular, these rule sets satisfy absorption of cut, contraction and congruence [PS10, §5].

Lemma 6.18. *If the rule set \mathcal{R} absorbs contraction and cut then \mathcal{R} also absorbs multicut.*

By Theorem 6.16 and Lemma 6.18, we obtain the following.

Corollary 6.19. *Suppose that \mathcal{R} absorbs contraction and cut. Then (Cut) is also admissible in \mathcal{SR} .*

As an immediate corollary, we obtain completeness of the cut-free calculus assuming that \mathcal{R} is *strongly* one-step complete:

Corollary 6.20. *Suppose that \mathcal{R} is one-step sound and strongly one-step complete. Then $\models \Gamma \Rightarrow \Delta$ iff $\mathcal{SR} \vdash \Gamma \Rightarrow \Delta$.*

Proof. This follows from Theorem 6.16 with the help of Proposition 5.11 and 5.12 of [PS10], the latter asserting precisely the absorption of cut and congruence. \square

The situation is more complex in presence of bounded operators where completeness of the Hilbert calculus is only guaranteed in presence of BDPL, and completeness of the associated sequent calculus relies on Paste_i^k . The difficulty in a proof of cut-elimination is a cut-end derivation where a cut is performed on $x \heartsuit [y_1 : \phi_1] \dots [y_n : \phi_n]$ which is introduced by Paste_i^k and a (one-step) rule where the same formula is principal. We leave this as an open problem:

Problem 6.21. Is there a way to modify the rules of \mathcal{BSR} so that completeness with respect to \mathcal{BHR} holds and cut is admissible?

7. CONCLUSIONS AND FURTHER WORK

We have introduced *coalgebraic predicate logic*, a natural first-order formalism that incorporates coalgebraic modalities and thus serves as an expressive language for coalgebras. As instances, it subsumes both standard relational first-order logic and Chang's first-order logic of neighbourhood systems [Cha73]; other instances include a first-order logic of non-monotone conditionals as well as first-order logics of integer-weighted relations that include weighted or (positive) Presburger modalities. We have shown completeness of two generic deduction systems, one phrased as a Hilbert system and the other as a sequent system. Moreover, we have developed the beginnings of a coalgebraic model theory.

In terms of future research, a promising avenue appears to be coalgebraic finite model theory; in fact, the first result in this direction is the existing finite version of the coalgebraic van Benthem-Rosen theorem [SP10a, LPSS12, SPLar]. It is worth observing that van Benthem-Rosen is a rare instance of a model-theoretic characterization of a fragment of first-order predicate logic that remains valid over finite models. The only other major result of this type we are aware of is the characterization of existential-positive formulas as exactly those preserved under homomorphisms [Ros08]. The result is relevant to constraint satisfaction problems and to database theory, as existential-positive formulas correspond to unions of conjunctive queries. Interestingly, the proof of Rossman’s result relies on Gaifman graphs, which also play a central role in the proof of the coalgebraic Rosen theorem.

Possible directions in coalgebraic model theory over unrestricted models include generalizations of standard results of classical model theory like Beth definability or interpolation and the Keisler-Shelah characterization theorem.

It remains to be seen which results of *modal model theory* building upon the interplay between modal and predicate languages can be generalized. Specific potential examples include Sahlqvist-type results for suitably well-behaved structures and analogues of results by Fine (does elementary generation imply canonicity, at least wherever the coalgebraic Jónsson-Tarski theorem [KKP05] obtains?) or Hodkinson [Hod06] (is there an algorithm generating a CML axiomatization for CPL-definable classes of coalgebras?).

Finally, a natural direction of investigation will be to study models based on coalgebras for endofunctors on categories other than **Set** and corresponding variants of CPL with non-boolean propositional bases.

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